



Geometrically-exact, intrinsic theory for dynamics of moving composite plates

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ABSTRACT

A geometrically-exact and fully intrinsic theory is presented for dynamics of composite plates undergoing large deformation. To say that the formulation is intrinsic means that it is without displacement and rotation variables. Although the equations are geometrically-exact, the highest degree nonlinearities are quadratic; there are no singularities associated with finite rotation. Methods for posing problems in this framework along with advantages of the formulation are discussed.

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1. Introduction

Whenever a theory is formulated which is independent of any displacement or rotational variables, it is frequently referred to as an “intrinsic” theory. The earliest development of plates of an intrinsic nature is the work of [Synge and Chien \(1941\)](#). Therein, with the aid of series expansions in powers of a small thickness parameter, approximate theories are derived from three-dimensional elasticity. A more recent intrinsic development is the work of [Simo and Fox \(1989a,b\)](#). They formulated a shell model in which finite deflections and rotations are taken into account. Their formulation regards the shell as a Cosserat surface, with an extensible director in the latter work, which is used in lieu of enforcing the usual plane stress assumption; thus, three-dimensional constitutive models can be employed, as done by [Koiter and Simmonds \(1972\)](#). Some intrinsic theories are strictly two-dimensional. That is, they provide no information about the three-dimensional behavior of the plate. Furthermore, they may rely on a two-dimensional constitutive law but not consider how to obtain such a law. For example, [Reissner \(1974\)](#) developed linear and nonlinear two-dimensional theories of shells in this manner. [Naghdi \(1972\)](#) provides an extensive review of this kind of approach, in which the constitutive relations are deliberately left out of consideration because they are unnecessary for the description of strain and the derivation of equilibrium equations. Other more recent developments for static behavior of shells are reviewed by [Hodges et al. \(1993\)](#).

Recently, a geometrically-exact, fully intrinsic theory for *dynamics* of curved and twisted composite beams was developed by [Hodges \(2003\)](#) having neither displacement nor rotation variables appearing in the formulation. There are 24 unknowns in the formulation including generalized strains, stress resultants, velocities, angular velocities, linear momenta, and angular momenta. The time history of these unknowns can be obtained from a system of algebraic-differential equations with spatial

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and time derivatives of the unknowns no higher than the first, and nonlinearities no higher than quadratic. When applied loads are independent of deformation, neither displacement nor rotation variables are needed for solving the equations. Displacement or rotation variables are only needed for recovery if such information is specifically required by the problem. In addition to the striking mathematical beauty of such a theory, the advantages include its being geometrically-exact, with nonlinearities no higher than quadratic, enhanced computational efficiency because lower-order Gaussian quadrature can be used, and the avoidance of singularities associated with rotation variables.

The precursor of this work by Hodges (1990) was originally developed for high-fidelity modeling of composite helicopter rotor blades. Indeed, the geometrically nonlinear problem of dynamics of moving composite beams has been solved by this theory and the cross-sectional analysis provided by the computer program VABS (see, for example, the work of Yu et al. (2002c)). However, there are still many structural components which cannot and should not be modeled as beams. To avoid the expense associated with full-blown three-dimensional (3D) formulation, engineers usually model components with thickness much smaller than the in-plane dimensions as shells; plates can be considered as degenerated shells with zero initial curvatures. Essentially, plate/shell models enable one to describe a flexible surface using only two coordinates with appropriate plate/shell elastic constants, such as those provided by the computer program VAPAS developed by Yu et al. (2002a,b). As more and more plate/shell-type components, particularly those of aircraft and spacecraft, become composite and highly flexible, the need for extending what has been done for composite beams, creating a geometrically-exact and intrinsic treatment for dynamics of moving composite plates/shells, becomes more apparent. To the best of the authors' knowledge this has not been done in the literature.

Although the focus of the present work is to develop a geometrically-exact, intrinsic theory for dynamics of moving composite plates, it is worthwhile to emphasize that the difficulty associated with composite materials has been adequately dealt with in the through-the-thickness analysis empowered by the computer program VAPAS; see Yu et al. (2002a,b). The through-the-thickness analysis enables a rigorous dimensional reduction from the original 3D nonlinear analysis into a 2D nonlinear analysis based on the variational asymptotic method (VAM) developed by Berdichevsky (1979). This allows the 2D plate analysis to be formulated exactly as a general continuum and confines all approximations to the through-the-thickness analysis, whose accuracy is guaranteed to be the best by VAM.

It is clear that the approach adopted by the present work is dramatically different from common approaches in the literature. For example, Prof. Librescu and co-workers have conducted a series of theoretical developments in the area of geometrically nonlinear analysis of anisotropic laminated plates/shells starting with Librescu (1987). The theory of Librescu and Schmidt (1988) presents an integrated geometrically nonlinear theory for the response of anisotropic laminated shells and is valid for small strains and moderate rotation. It includes the effects of transverse shear and transverse normal deformation. By estimating the order of magnitude of linearized strain and rotation components based on a higher-order representation of the displacement field through the shell thickness, appropriate strain-measures are obtained. The theory incorporates as special cases, other non-linear shell and plate theories. Librescu and Stein (1991) presents a geometrically nonlinear theory like the von Karman theory with generalizations to anisotropic material and as well as inclusion of transverse shear and normal deformation. The theory is applied to the post-buckling analysis of flat composite panels which are symmetrically laminated from transversely isotropic layers. The work done by Prof. Librescu is restricted to moderate rotations and is quite complicated in its final form because it is written in terms of displacements.

To best illustrate the theory and for the sake of simplicity, we are going to focus on modeling of plates as 2D continua with appropriate kinematics, constitutive relations, and equations of motion. After obtaining the full set of intrinsic equations, we will briefly discuss solution procedures and possible applications of these equations. This theory can be straightforwardly extended to model shells. We are still exploring the advantages and application of this unique theory for predicting structural behavior of plates and shells.

2. Kinematics

The equations of 2D plate theory are written over the reference plane which is described by two coordinates, x_α . (Here and throughout the paper Latin indices assume 1, 2, 3; and Greek indices assume values 1 and 2. Dummy indices are summed over their range except where explicitly indicated.) As sketched in Fig. 1, every point on the reference plane can be represented by a position vector \mathbf{r} in the undeformed state and \mathbf{R} in the deformed state with respect to a fixed point O in inertial space. To facilitate derivation, we introduce two orthonormal triads \mathbf{b}_i and \mathbf{B}_i , for the undeformed and deformed state, respectively. We choose \mathbf{b}_α to be the base vectors associated with x_α and $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2$, and \mathbf{B}_i is coincident with \mathbf{b}_i when the structure is undeformed. With no limitation on the rotation of any material element on the reference plane during deformation, one can use the direction cosine matrix to relate these two triads as:

$$\mathbf{B}_i = C_{ij} \mathbf{b}_j \quad C_{ij} = \mathbf{B}_i \cdot \mathbf{b}_j \quad (1)$$

Although various schemes can be used to parameterize the direction rotation matrix, they are irrelevant because the rotation variables are found to be unnecessary in the fully intrinsic theory developed here.

As a crucial step in the derivation of an intrinsic formulation, following Hodges et al. (1993), we define the 2D generalized strains as

$$\mathbf{R}_{,\alpha} = (\mathbf{B}_\alpha + \epsilon_{\alpha\beta} \mathbf{B}_\beta + 2\gamma_{\alpha 3} \mathbf{B}_3) \quad (2)$$

$$\mathbf{B}_{i,\alpha} = (-K_{\alpha 2} \mathbf{B}_1 + K_{\alpha 1} \mathbf{B}_2 + K_{\alpha 3} \mathbf{B}_3) \times \mathbf{B}_i \quad (3)$$

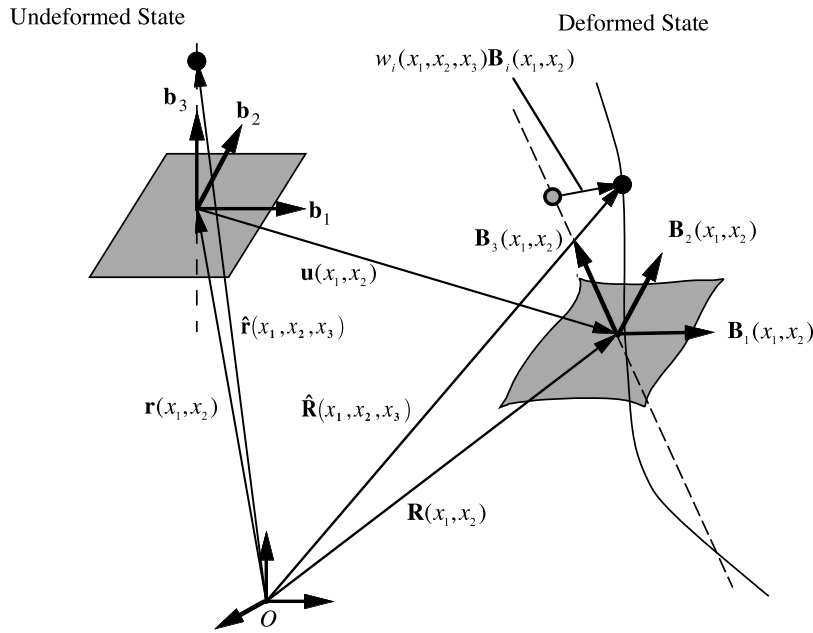


Fig. 1. Schematic of plate deformation.

where $\epsilon_{\alpha\beta}$ are the in-plane generalized strains, $\gamma_{\alpha 3}$ are the transverse shear generalized strains, and $K_{\alpha j}$ are the curvatures of the deformed surface. To ensure a unique mapping between the original 3D formulation and the reduced 2D formulation, the rotation of \mathbf{B}_α around \mathbf{B}_3 was constrained in the work of Yu et al. (2002b) such that

$$\mathbf{B}_1 \cdot \mathbf{R}_{,2} = \mathbf{B}_2 \cdot \mathbf{R}_{,1} \tag{4}$$

which leads to symmetric in-plane generalized strains, i.e.

$$\epsilon_{12} = \epsilon_{21} \tag{5}$$

For convenience of manipulation using matrix algebra, we can arrange the strain measures in column matrices as

$$\gamma_\alpha = \begin{Bmatrix} \epsilon_{\alpha 1} \\ \epsilon_{\alpha 2} \\ 2\gamma_{\alpha 3} \end{Bmatrix} \quad K_\alpha = \begin{Bmatrix} -K_{\alpha 2} \\ K_{\alpha 1} \\ K_{\alpha 3} \end{Bmatrix} \tag{6}$$

It has been shown by Hodges et al. (1993) that these 11 strains [recalling the identity of in-plane shear in Eq. (5)] are related by six compatibility equations, which means only five independent displacement/rotation variables exist for the deformation of the reference surface. If we choose u to represent the displacement vector of any point on the deformed reference surface and C to represent the finite rotation as defined in Eq. (1), it can be easily shown that the following kinematic relations hold

$$\gamma_\alpha = C(e_\alpha + u_{,\alpha}) - e_\alpha \tag{7}$$

$$\widetilde{K}_\alpha = -C_{,\alpha} C^T \tag{8}$$

where $(\widetilde{\cdot})_{ij} = -e_{ijk}(\cdot)_{,k}$, $e_1 = [100]^T$, and $e_2 = [010]^T$.

3. Intrinsic equations of motion

Hamilton’s extended principle for the surface can be written as

$$\int_{t_1}^{t_2} \int_s [\delta(\mathcal{K} - \mathcal{U}) + \overline{\delta\mathcal{W}}] ds dt = \overline{\delta\mathcal{A}} \tag{9}$$

where s denotes the reference surface, t_α are arbitrary fixed times, \mathcal{K} and \mathcal{U} are the kinetic and strain energy per unit area, respectively, $\overline{\delta\mathcal{W}}$ is the virtual work of applied loads per unit area, and $\overline{\delta\mathcal{A}}$ is the virtual action at the boundary of plate and at the ends of the time interval. The bars on the last two quantities indicate that they may not be variations of any functionals, and the last term only affects boundary and initial conditions. The original 3D model can be rigorously reduced to a 2D model using the variational asymptotic method of Berdichevsky (1979). The resulting strain energy per unit area can be expressed as a quadratic form such that

$$\mathcal{U} = \mathcal{U}(\epsilon_{11}, 2\epsilon_{12}, \epsilon_{22}, 2\gamma_{13}, 2\gamma_{23}, K_{11}, K_{12} + K_{21}, K_{22}) \tag{10}$$

the variation of which can be obtained as

$$\begin{aligned} \delta \mathcal{U} &= \frac{\partial \mathcal{U}}{\partial \epsilon_{11}} \delta \epsilon_{11} + \frac{\partial \mathcal{U}}{\partial (2\epsilon_{12})} \delta (2\epsilon_{12}) + \frac{\partial \mathcal{U}}{\partial \epsilon_{22}} \delta \epsilon_{22} + \frac{\partial \mathcal{U}}{\partial (2\gamma_{13})} \delta (2\gamma_{13}) + \frac{\partial \mathcal{U}}{\partial (2\gamma_{23})} \delta (2\gamma_{23}) \\ &+ \frac{\partial \mathcal{U}}{\partial K_{11}} \delta K_{11} + \frac{\partial \mathcal{U}}{\partial (K_{12} + K_{21})} \delta (K_{12} + K_{21}) + \frac{\partial \mathcal{U}}{\partial K_{22}} \delta K_{22} \\ &\equiv N_{11} \delta \epsilon_{11} + N_{12} \delta (2\epsilon_{12}) + N_{22} \delta \epsilon_{22} + Q_1 \delta (2\gamma_{13}) + Q_2 \delta (2\gamma_{23}) \\ &+ M_{11} \delta K_{11} + M_{12} \delta (K_{12} + K_{21}) + M_{22} \delta K_{22} \end{aligned} \tag{11}$$

where $N_{\alpha\beta}$ are generalized in-plane forces, Q_α are generalized shear forces, and $M_{\alpha\beta}$ are generalized moments. The constitutive relation between the generalized strains and generalized forces/moments is obtained from VAPAS in the form

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \\ K_{11} \\ K_{22} \\ K_{12} + K_{21} \end{Bmatrix} + \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \tag{12}$$

and

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} \begin{Bmatrix} 2\gamma_{13} \\ 2\gamma_{23} \end{Bmatrix} \tag{13}$$

where F_1, F_2, \dots, F_6 are extracted from the work done on the warping displacement field by applied surface tractions and body forces; see Yu et al. (2002a,b). It should be noted that the asymptotic analysis presented in these references and embedded in VAPAS shows that K_{12} and K_{21} appear only in the combination $K_{12} + K_{21}$ in the strain energy. The difference will show up in higher-order theories; however, asymptotic theories of such order have not been published to date. In-plane curvatures $K_{\alpha 3}$ will not appear in the strain energy of any theory based solely on 3D elasticity; a 3D foundation based on a micro-polar or couple-stress formulation, such as put forth by Reissner (1972, 1973), would be required to elicit these terms in an asymptotically derived 2D theory.

As shown by Hodges et al. (1993), the variations of the strains can be obtained in terms of virtual displacements and virtual rotations as

$$\begin{aligned} \delta \epsilon_{\alpha\beta} &= e_{\beta}^T [\delta \bar{q}_{,\alpha} + \widetilde{K}_\alpha \delta \bar{q}] + (\widetilde{e}_\alpha + \widetilde{\gamma}_\alpha) \delta \bar{\psi} \\ \delta (2\gamma_{\alpha 3}) &= \delta \bar{q}_{3,\alpha} + \delta \bar{\psi}_\alpha + \epsilon_{\alpha\beta} \delta \bar{\psi}_\beta - K_{\alpha\beta} \delta \bar{q}_\beta \\ \delta K_\alpha &= \delta \bar{\psi}_{,\alpha} + \widetilde{K}_\alpha \delta \bar{\psi} \end{aligned} \tag{14}$$

where $\delta \bar{q}_i$ and $\delta \bar{\psi}_i$ are measure numbers of the virtual displacement and virtual rotation, respectively, in the basis \mathbf{B}_i . If there are body forces and surface tractions applied to the plate, the virtual work can be obtained as

$$\delta \mathcal{W} = \delta \bar{q}_i f_i + \delta \bar{\psi}_\alpha m_\alpha \tag{15}$$

where f_i and m_α are calculated from the applied tractions on the surfaces, and body forces in the modeling process; see Yu et al. (2002a).

Having obtained $\delta \mathcal{U}$ and $\delta \mathcal{W}$, our next step is to express the kinetic energy, \mathcal{K} , and its variation in intrinsic form. By definition, the kinetic energy of the plate can be written as

$$\mathcal{K} = \frac{1}{2} \langle \rho \mathbf{v}^{MI} \cdot \mathbf{v}^{MI} \rangle = \frac{1}{2} \langle \rho (v_B^{MI})^T v_B^{MI} \rangle \tag{16}$$

where the bracket denotes integration through the thickness, ρ is the mass density, and \mathbf{v}^{MI} is the inertial velocity of any material point in the moving plate. For low-frequency dynamics it is appropriate to ignore the warping in the kinetic energy. Thus, the column matrix of measure numbers of the velocity, expressed in the basis \mathbf{B}_i , is given by

$$v_B^{MI} = V + \widetilde{\Omega} \xi \tag{17}$$

where $\xi = [00x_3]^T$ with x_3 as the normal coordinate, V is the column matrix of inertial velocity measures of any material point on the plate reference surface, and Ω is the column matrix of inertial angular velocity measures, both expressed in the basis of \mathbf{B}_i such that

$$\begin{aligned} V &= C(v + \dot{u} + \tilde{\omega}u) \\ \widetilde{\Omega} &= -\dot{C}C^T + C\tilde{\omega}C^T \end{aligned} \tag{18}$$

with column matrices v and ω containing the measure numbers in the \mathbf{b}_i basis of the inertial velocity vector of any point on the undeformed reference plane and the inertial angular velocity of \mathbf{b}_i , respectively. Following the procedure spelled out by

Hodges (2003), we can differentiate Eqs. (18) with respect to x_α , and differentiate Eqs. (7) and (8) with respect to time, to obtain a set of intrinsic kinematical partial differential equations

$$\begin{aligned} V_{,\alpha} &= \dot{\gamma}_\alpha + \tilde{V}K_\alpha + \tilde{\Omega}(e_\alpha + \gamma_\alpha) \\ \Omega_{,\alpha} &= \dot{K}_\alpha + \tilde{\Omega}K_\alpha \end{aligned} \tag{19}$$

without using displacement and rotation variables.

Introducing inertial constants commonly used in plate dynamics

$$\mu = \langle \rho \rangle \quad \mu \tilde{\zeta} = [00 \langle x_3 \rho \rangle]^T \quad \mu r^2 = \langle x_3^2 \rho \rangle \tag{20}$$

we can express the variation of kinetic energy per unit area as

$$\delta \mathcal{K} = \delta V^T P + \delta \Omega^T H \tag{21}$$

with P and H as the linear and angular momenta, respectively, given by

$$\begin{Bmatrix} P \\ H \end{Bmatrix} = \begin{bmatrix} \mu \Delta & -\mu \tilde{\zeta} \\ \mu \tilde{\zeta} & I \end{bmatrix} \begin{Bmatrix} V \\ \Omega \end{Bmatrix} \tag{22}$$

where

$$I = \begin{bmatrix} \mu r^2 & 0 & 0 \\ 0 & \mu r^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{23}$$

It is easy to infer that $H_3 = 0$. Because of the arrangement of the curvature vectors in Eq. (6), Ω and H are arranged accordingly such that

$$\Omega = \begin{Bmatrix} -\Omega_2 \\ \Omega_1 \\ \Omega_3 \end{Bmatrix} \quad H = \begin{Bmatrix} -H_2 \\ H_1 \\ H_3 \end{Bmatrix} \tag{24}$$

In Eq. (21), δV and $\delta \Omega$ can be easily obtained following the derivation of Hodges (1990), with the results given by

$$\begin{aligned} \delta V &= \delta \bar{q} + \tilde{\Omega} \delta \bar{q} + \tilde{V} \delta \bar{\psi} \\ \delta \Omega &= \delta \bar{\psi} + \tilde{\Omega} \delta \bar{\psi} \end{aligned} \tag{25}$$

To complete the formulation, we also need to express the virtual action $\delta \bar{\mathcal{A}}$ along the boundary of plate and at the ends of the time interval in terms of virtual displacements and virtual rotations. Along the boundary one can specify appropriate combinations of displacements, rotations (geometrical boundary conditions), and running forces and moments (natural boundary conditions) along the boundary around the reference plane. Suppose on boundary Γ (see Fig. 2), we specify a force resultant $\hat{N}_{v\tau}$ and a moment resultant $\hat{M}_{v\tau}$ along the outward normal of the boundary curve tangent to the reference surface v , $\hat{N}_{v\tau}$ and $\hat{M}_{v\tau}$ along the tangent of the boundary curve τ , \hat{N}_{v3} along the normal of the reference surface. Also, we assume that at the ends of the time interval, we have virtual actions $(\delta \bar{q}^T \hat{P} + \delta \bar{\psi}_\alpha \hat{H}_\alpha)$ entering and leaving the system. Then $\delta \bar{\mathcal{A}}$ can be expressed as:

$$\delta \bar{\mathcal{A}} = \int_s (\delta \bar{q}^T \hat{P} + \delta \bar{\psi}_\alpha \hat{H}_\alpha) \Big|_{t_1}^{t_2} ds - \int_{t_1}^{t_2} \int_\Gamma (\hat{N}_{v\tau} \delta \bar{q}_v + \hat{N}_{v\tau} \delta \bar{q}_\tau + \hat{N}_{v3} \delta \bar{q}_3 + \hat{M}_{v\tau} \delta \bar{\psi}_v + \hat{M}_{v\tau} \delta \bar{\psi}_\tau) d\Gamma dt = 0 \tag{26}$$

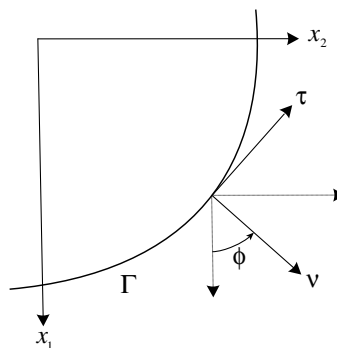


Fig. 2. Schematic of an arbitrary boundary.

Here subscripts ν and τ are used to indicate direction and should not be treated as indices. Hence, the summation convention is not applicable to these subscripts.

Making use of Eqs. (21) and (25) along with Eqs. (11), (14), and (15), we write the partial differential equations of motion from Eq. (9) as

$$\begin{aligned} N_{11,1} + (N_{12} + \mathcal{N})_{,2} - K_{13}(N_{12} - \mathcal{N}) - K_{23}N_{22} + Q_1K_{11} + Q_2K_{21} + f_1 &= \dot{P}_1 + \Omega_1P_3 - \Omega_3P_2 \\ N_{22,2} + (N_{12} - \mathcal{N})_{,1} + K_{23}(N_{12} + \mathcal{N}) + K_{13}N_{11} + Q_1K_{12} + Q_2K_{22} + f_2 &= \dot{P}_2 + \Omega_3P_1 + \Omega_2P_3 \\ Q_{1,1} + Q_{2,2} - K_{11}N_{11} - K_{22}N_{22} - (K_{12} + K_{21})N_{12} + (K_{12} - K_{21})\mathcal{N} + f_3 &= \dot{P}_3 - \Omega_2P_2 - \Omega_1P_1 \\ M_{11,1} + M_{12,2} - Q_1(1 + \epsilon_{11}) - Q_2\epsilon_{12} + 2\gamma_{13}N_{11} + 2\gamma_{23}(N_{12} + \mathcal{N}) - M_{12}K_{13} - M_{22}K_{23} + m_1 &= \dot{H}_1 - \Omega_3H_2 - V_1P_3 + V_3P_1 \\ M_{12,1} + M_{22,2} - Q_1\epsilon_{12} - Q_2(1 + \epsilon_{22}) + 2\gamma_{13}(N_{12} - \mathcal{N}) + 2\gamma_{23}N_{22} + M_{11}K_{13} + M_{12}K_{23} + m_2 &= \dot{H}_2 + \Omega_3H_1 - V_2P_3 + V_3P_2 \end{aligned} \quad (27)$$

where

$$\begin{aligned} (2 + \epsilon_{11} + \epsilon_{22})\mathcal{N} &= (N_{22} - N_{11})\epsilon_{12} + N_{12}(\epsilon_{11} - \epsilon_{22}) + M_{22}K_{21} - M_{11}K_{12} + M_{12}(K_{11} - K_{22}) - \Omega_1H_2 + \Omega_2H_1 - V_1P_2 \\ &\quad + V_2P_1 \end{aligned} \quad (28)$$

The associated natural boundary conditions on Γ are

$$\begin{aligned} \hat{N}_{\nu\nu} &= n_1^2N_{11} + 2n_1n_2N_{12} + n_2^2N_{22} \\ \hat{N}_{\nu\tau} &= n_1n_2(N_{22} - N_{11}) + (n_1^2 - n_2^2)N_{12} - \mathcal{N} \\ \hat{N}_{\nu 3} &= n_1Q_1 + n_2Q_2 \\ \hat{M}_{\nu\nu} &= n_1^2M_{11} + 2n_1n_2M_{12} + n_2^2M_{22} \\ \hat{M}_{\nu\tau} &= n_1n_2(M_{22} - M_{11}) + (n_1^2 - n_2^2)M_{12} \end{aligned} \quad (29)$$

where \mathcal{N} is a Lagrange multiplier to enforce Eq. (5), $n_1 = \cos \phi$, $n_2 = \sin \phi$, and ϕ is the angle between the outward normal of the boundary and x_1 direction as shown in Fig. 2. Again ν and τ are subscripts and no summation should be applied even when they are repeated in Eqs. (29). If the generalized forces are not specified along the boundary, the corresponding geometric boundary conditions should be specified, but without displacement and rotation variables they must be posed in terms of velocities and angular velocities on the boundary. Finally, for the conditions at the ends of time intervals, one can either prescribe the 2D displacement field, so that the 2D generalized strains are known, or corresponding momenta such that

$$\begin{aligned} P(t_1) &= \hat{P}(t_1) \quad H_x(t_1) = \hat{H}_x(t_1) \\ P(t_2) &= \hat{P}(t_2) \quad H_x(t_2) = \hat{H}_x(t_2) \end{aligned} \quad (30)$$

4. Potential solution strategies

The intrinsic formulation derived above is complete and can be solved. The formulation has a total of 31 equations which include five equations of motion [Eqs. (27)], one constraint equation [Eq. (28)], 12 generalized strain–velocity equations [Eqs. (19)], five inertial constitutive equations relating the generalized velocity to generalized momenta [Eqs. (22)], and a total of eight structural constitutive equations relating stress resultants to generalized strains [Eqs. (12) and (13)]. These equations can be used to solve for the 31 variables which include 11 generalized strain measures [not 12 – recall Eq. (5)], eight stress resultants, three velocities, three angular velocities, three linear momenta, two angular momenta ($H_3 = 0$), and \mathcal{N} .

The inertial and structural constitutive equations [Eqs. (22), (12) and (13)] are linear algebraic relations. While solving the problem, the constitutive equations may be used to replace some of the variables in terms of others. We could replace the stress resultants by their expression in terms of the strains measures and replace the generalized momenta by their expression in terms of the six generalized velocities (i.e. the three velocities and three angular velocities). Thus, we can write the complete formulation in terms of only 18 unknowns (11 generalized strains, three velocities, three angular velocities, and \mathcal{N}) which would be solved using 18 equations [Eqs. (19), (27) and (28)].

Note that unless the strain measures are calculated by differentiating a continuous displacement field, we have to satisfy six compatibility equations in addition to the symmetric in-plane generalized shear strain constraint. The compatibility equations of Hodges et al. (1993) can be written in matrix form as

$$\begin{aligned} E_1 &= \gamma_{1,2} - \gamma_{2,1} - \tilde{K}_1(\gamma_2 + e_2) + \tilde{K}_2(\gamma_1 + e_1) = 0 \\ E_2 &= K_{1,2} - K_{2,1} - \tilde{K}_1K_2 = 0 \end{aligned} \quad (31)$$

Because we do not have displacement variables in the formulation, we need to examine the possible role of these static compatibility equations. Using the 12 generalized strain–velocity equations [Eqs. (19)] and the continuity of the velocity field ($V_{i,\alpha\beta} = V_{i,\beta\alpha}$ and $\Omega_{i,\alpha\beta} = \Omega_{i,\beta\alpha}$), we can also find dynamic compatibility equations of the form

$$\begin{aligned}\dot{E}_1 + \tilde{\Omega}E_1 + \tilde{V}E_2 &= 0 \\ \dot{E}_2 + \tilde{\Omega}E_2 &= 0\end{aligned}\quad (32)$$

Thus, when we satisfy the 12 generalized strain–velocity relations, Eqs. (19), we implicitly satisfy these dynamic compatibility equations. Thus, (i) if we start with a compatible generalized strain field ($E_1 = E_2 = 0$), then the generalized strain field will remain compatible for later times; and (ii) if we have a static problem ($\dot{E}_1 = \dot{E}_2 = V = \Omega = 0$), then the dynamic compatibility equations are trivially satisfied, but the static ones must be enforced.

The solution strategy is then different for static and dynamic cases. For the dynamic case, compatibility equations are not necessary because we are including the kinematical equations that relate the rate of change of the generalized strain measures to generalized velocities [Eqs. (19)] in the formulation. The compatibility equations are thus included in the kinematical partial differential equations in time-differentiated form. It should be noted that one must start with a compatible generalized strain field, i.e. the initial conditions must satisfy the static compatibility conditions.

For the static problem, we can write the complete formulation in terms of only 12 unknowns (11 generalized strains and \mathcal{N}). The six generalized velocities are identically zero and the 12 generalized strain–velocity kinematic equations are exactly satisfied by the six generalized velocity variables. Since we lose 12 generalized strain–velocity equations [Eqs. (19)], we need the six (static) compatibility equations in addition to the five equations of motion (27) plus Eq. (28) to form a complete solvable set.

Finally, it may be of interest to have the same set of equations to solve both the dynamic as well as static problem. To form such a set, we need to pick six of the generalized strain–velocity equations which when complemented by the six compatibility equations can form a complete set. There are number of ways to choose six generalized strain–velocity equations from the set of 12, but the ones that make the most physical and mathematical sense are the kinematic relations for $\dot{\epsilon}_{11}$, $\dot{\epsilon}_{22}$, $2\dot{\gamma}_{13}$, $2\dot{\gamma}_{23}$, $(\dot{K}_{12} + \dot{K}_{21})$ and $\dot{\epsilon}_{12} = \dot{\epsilon}_{21}$. The fifth equation is the addition of the two corresponding kinematic relations and the sixth equation uses the two corresponding dynamic kinematic equations to give a static constraint (no time derivative) on Ω_3 (similar to the displacement constraint on ϕ_3) in Hodges et al. (1993) Therefore, we can solve for the 18 variables by using a set of 18 equations composed of five equations of motion [Eqs. (27)], one constraint equation [Eq. (28)], six of the 12 generalized strain–velocity equations [Eqs. (19)] selected as described above and the six static compatibility equations [Eqs. (31)]. This set of equations has an additional advantage in that only ten of these equations are dynamic while the other eight are static constraints. Thus, the equations are consistent with the fact that the plate has only five independent displacement/rotation variables.

5. Concluding remarks

A fully intrinsic formulation, i.e. devoid of displacement and rotation variables, for the dynamics of a moving composite plate has been presented. Although the equations are geometrically-exact, the highest degree nonlinearities are quadratic; there are no singularities associated with finite rotation. Appropriate solution methodology will be presented in a later paper. One could use either a finite-element formulation or Galerkin's method. In either case, because of the low-degree nonlinearities, the resulting integrals can be calculated with reduced effort. Finally, it is yet to be determined what value other than elegance this formulation may add to the solution of problems in structural dynamics.

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