Review Notes for Week 1

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1 Lecture # 1, August 27, 2001

Introduction to the course, the need for determining stresses and deflections, the division of the course into three parts (i) review of stress and strain, shear force and bending moment diagrams, and yielding of material, (ii) Thin-walled stressed skin structures, (iii) and finite element analysis.

Review of stress in an axial bar ($\sigma = P/A$) subjected to a load. Normal and shear stresses on an arbitrary plane for 1-D case. We discussed that the stress, a measure of force intensity, is a point function, *i.e.* its value varies from point to point. Furthermore, since every plane passing through the point of interest will have different force exerted on it by the rest of the body, the stress also depends on the orientation of the plane of interest passing through that point.

We started to talk about the state of stress at a point in a 3-D body subjected to arbitrary loads.

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2 Lecture # 2, August 29, 2001

The lecture was devoted to describing the state of the stress at a point. The stress acting at a point will depend upon the orientation, described by the unit normal, say $\vec{n}$, of the plane of interest passing through that point.

Let δA is the area of an infinitesimal plane, with unit normal $\vec{n}$, enclosing the point of interest and let $\delta\vec{P}_n$ is the force vector exerted by the rest of the body on the enclosing plane. Note that, in general, the force $\delta\vec{P}_n$, will neither be normal nor tangent to the plane. The average stress vector on this area can be obtained by divididing the force vector $\delta\vec{P}_n$ with area δA. As we continuously reduce the value of the area δA, we achieve the limiting value of the stress vector.

This limiting value is the value of the stress vector at the point of interest acting on a plane with unit normal $\vec{n}$.

The *stress vector* $\vec{\sigma}_n$ is thus defined as:

$$\vec{\sigma}_n = \lim_{\delta A \to 0} \frac{\delta\vec{P}_n}{\delta A}$$

We can resolve the stress vector $\vec{\sigma}_n$ in the directions, both normal and tangent, to the plane, as:

$$\begin{aligned} \sigma_{nn} &= \vec{\sigma}_n.\vec{n} \ \ \text{(normal component)} \\ \tau_{nt} &= \sqrt{(||\vec{\sigma}_n||^2 - \sigma_{nn}^2)} \ \ \text{(tangential component)} \end{aligned}$$

Here $||\vec{\sigma}_n||$ is the magnitude of the normal stress vector σ_{nn}. The stress vector at a point thus varies with the orientation, given by the unit normal $\vec{n}$, of the plane under consideration.

Does it, then, mean that we need to determine the stress vectors for all (**infinite!**) planes passing through the point of interest. Of course, the answer is no. We only need to find the stress vector on any three mutually orthogonal planes. The stress vector on any arbitrary plane, whose unit normal is known, passing through that point can then be obtained from the three stress vectors using Statics (equilibrium of **forces** in all the three directions) and the components of the unit normal that characterises the given plane.

Let x, y, and z be a co-ordinate system. We can, then, determine the stress vectors acting on three mutually orthogonal planes described by the unit base vectors, $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$. Consider a differential area δA_x with a unit normal $\vec{\imath}$, *i.e.* a plane normal to the x-axis, also called the x-plane. Let $\delta\vec{F}_x$ is the force vector exerted by the rest of the body on the enclosing plane. The stress vector $\vec{\sigma}_x$ is defined as:

$$\begin{aligned}\vec{\sigma}_x &= \lim_{\delta A_x \to 0} \frac{\delta \vec{F}_x}{\delta A_x} \\ &= \sigma_{xx}\vec{\imath} + \tau_{xy}\vec{\jmath} + \tau_{xz}\vec{k}\end{aligned}$$

The component σ_{xx}, along the x- direction, is called the *normal stress component* in the x-direction; the other two components, τ_{xy}, and τ_{xy} that act tangential to the plane are called *shear components.* Note that the *first* subscript, x in this case, represents the orientation of the plane on which the stress is being calculated and the *second* subscript represents the direction of the component. Thus τ_{xy} represents the stress component acting on the x-plane in the y- direction.

Similarly, we consider, separately, a differential area δA_y with a unit normal $\vec{\jmath}$, *i.e.* a plane normal to the y-axis, or the y-plane; and a differential area δA_z with a unit normal $\vec{k}$, *i.e.* a plane normal to the z-axis, or the z-plane. Let $\delta\vec{F}_y$ is the force vector exerted by rest of the body on the enclosing plane with unit vector $\vec{\jmath}$ and $\delta\vec{F}_z$ is the force vector exerted by rest of the body on the enclosing plane with unit vector $\vec{k}$. We can write the stress vectors $\vec{\sigma}_y$, and $\vec{\sigma}_z$, acting, respectively, on the y- and z-planes, as:

$$\begin{aligned}\vec{\sigma}_y &= \lim_{\delta A_y \to 0} \frac{\delta \vec{F}_y}{\delta A_y} \\ &= \tau_{yx}\vec{\imath} + \sigma_{yy}\vec{\jmath} + \tau_{yz}\vec{k} \\ \vec{\sigma}_z &= \lim_{\delta A_z \to 0} \frac{\delta \vec{F}_z}{\delta A_z} \\ &= \tau_{zx}\vec{\imath} + \tau_{zy}\vec{\jmath} + \sigma_{zz}\vec{k}\end{aligned}$$

Together, the three stress vectors, $\vec{\sigma}_x$, $\vec{\sigma}_y$, and $\vec{\sigma}_z$, describe the state of the stress at a given point. *We thus need a total of* **nine** *stress components to completely describe the state of stress at a given point.*

These nine components are often written in the matrix form, called *stress tensor*:

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

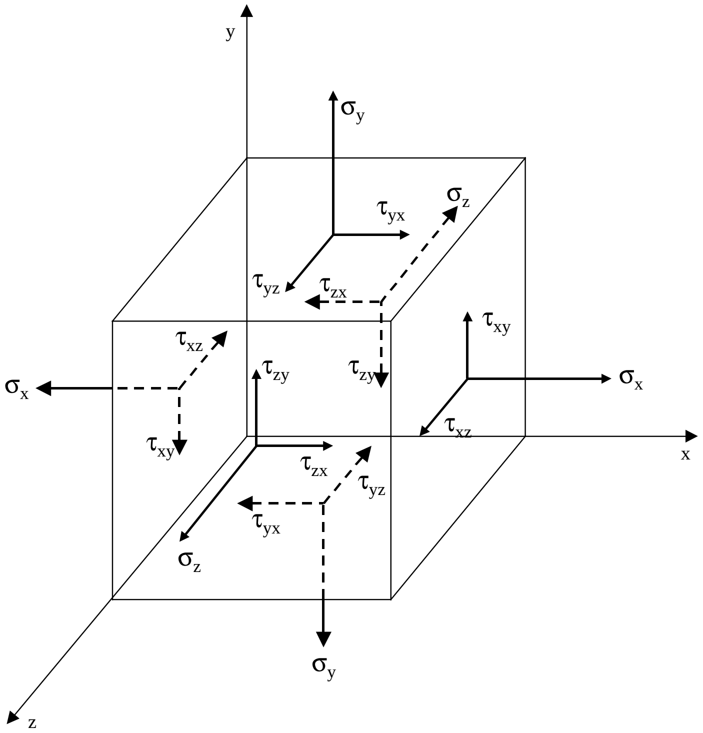

Figure 1: Nine Components of the State of Stress at a point. The positive direction for each of the stress component is shown here.

These components are graphically illustrated in the attached Figure 1.

Following **sign convention** is used in expressing various stress components: (1) A normal stress component is positive if it is in the direction of the outer normal, *i.e.* if it causes tension; (2) A shear component is positive if it acts in the positive direction of the respective co-ordinate when acting on a plane with a unit outer normal that is along the positive axis. For example, the shear stress component τ_{xy} is positive if it is in the direction of positive y direction on the face for which the unit outer normalis in the positive x-direction and it is in the negative direction of the $y-$ axis on the face for which the unit outer normal is in the negative x- direction.

Additional Reading: See pages 1–5 of your text (Megson) or pages 24–27 of Beer, Johnston and DeWolf (3rd Ed.).

In a subsequent lecture, we will see that, for a differential element to be in moment equilibrium, the stress tensor must be symmetric, *i.e.* $\tau_{xy} = \tau_{yx}$, $\tau_{xy} = \tau_{yx}$, and $\tau_{xy} = \tau_{yx}$ We thus need only six, not nine, independent components to completely describe the state of the stress.

3 Lecture # 3, August 31, 2001

Using an example, the idea of, determining stress vector on an arbitrary plane given the nine stress components and the direction cosines of the unit normal on that plane, was illustrated. The example solved was as follows:

The state of stress at a point in a component is given as:

$$\begin{bmatrix} 0 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 0 \end{bmatrix} \text{ MPa}$$

a. Show this state of stress on a differential element.

b. Determine the stress vectors and the total force vectors acting on the faces OAC, OBC, and OBA. Note that $OA = OB = OC = \Delta$ meters.

c. Determine the stress vector acting on the face ABC.

Hint I: The element OABC is in static equilibrium. The forces acting on face OABC should be in static equilibrium with the forces acting on the three mutually orthogonal faces OAC, OBC, and OBA.

Hint II: One important requirement for solving this problem will be to first determine the unit normal, say $\vec{n}$, and the area of the triangle ABC. One can determine both of these simultaneously by using the idea of *cross-product of two vectors*. The attached Fig. 2. shows vectors $\vec{OA}$, $\vec{OB}$, and the unit normal vector, $\vec{n}$, normal to both vectors $\vec{OA}$, and $\vec{OB}$. The three vectors, $\vec{OA}$, $\vec{OB}$, and $\vec{n}$, where $\vec{n}$, a unit vector, is normal to both $\vec{OA}$, and $\vec{OB}$; form a right-handed triplet. Recall a right-handed triplet is similar to a right-handed screw. As we rotate a right handed screw in anti-clock wise direction (*i.e.* rotate the vector $\vec{OA}$ towards $\vec{OB}$), the screw will move in the direction of $\vec{n}$. The cross-product of vectors $\vec{OA}$ and $\vec{OB}$ will be normal to both these vectors, thus it will be in the direction of unit normal $\vec{n}$. The desired cross-product, say $\vec{OC}$, can be expressed as:

$$\begin{aligned}\vec{OA}\,\mathrm{x}\,\vec{\mathrm{OB}} &= \vec{OC} \\ &= ||OA||\,||OB||\sin\phi\,\vec{n} \\ &= 2\,(\text{Area of Triangle}\,OAB)\,\vec{n} \qquad (1)\end{aligned}$$

Here ϕ is the angle between the two vectors. The cross-product of two vectors can also be written as:

$$\vec{OA}\,\mathrm{x}\,\vec{\mathrm{OB}} = \det\begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

Here a_1, b_1, and c_1 are the components of $\vec{OA}$ in x, y, and z directions, respectively; and similarly, a_2, b_2, and c_2 are the components of $\vec{OB}$ in x, y, and z directions, respectively. Here 'det' stands for the determinant of the matrix. The cross-product thus becomes:

$$\begin{aligned}\vec{OA}\,\mathrm{x}\,\vec{\mathrm{OB}} &= \vec{\imath}\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\ &= \underbrace{(b_1c_2 - b_2c_1)}_{a_3}\vec{\imath} + \underbrace{(a_2c_1 - a_1c_2)}_{b_3}\vec{\jmath} + \underbrace{(a_1b_2 - a_2b_1)}_{c_3}\vec{k} \qquad (2)\end{aligned}$$

Comparing Eq. 1 and Eq. 2, one can obtain both the unit normal and the area of the triangle, OAB. These are given as follows:

$$\begin{aligned}\vec{n} &= \frac{a_3\vec{\imath} + b_3\vec{\jmath} + c_3\vec{k}}{\sqrt{(a_3^2 + b_3^2 + c_3^2)}} \\ 2\mathrm{Area}_{OAB} &= \sqrt{(a_3^2 + b_3^2 + c_3^2)} \qquad (3)\end{aligned}$$

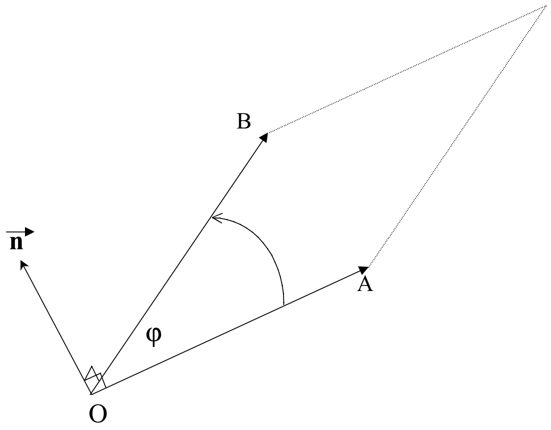

Figure 2: Cross-product of two vectors $\vec{OA}$, and $\vec{OB}$. The cross-product of two vectors is a vector which is normal to both the vectors and whose magnitude is equal to the area of the parellogram formed by the two vectors. Note that $\vec{OA}\,\mathrm{x}\,\vec{\mathrm{OB}} = -\vec{\mathrm{OB}}\,\mathrm{x}\,\vec{\mathrm{OA}}$