

# Review Notes for Week 2

Rakesh K. Kapania  
Professor, Aerospace and Ocean Engineering  
Virginia Polytechnic Institute and State University  
Blacksburg, VA 24061-0203\*

September 5, 2001

## 1 Lecture # 4, September 3, 2001

We spent most of the time on finishing the example on determining, given the nine stress components at a point, the stress vector acting on a plane passing through that point whose unit normal is known. Experience gained from the application of principles of Statics for a specific example was used to derive **Cauchy's** relation. To that end, consider an arbitrary plane  $ABC$ , whose unit normal is  $\vec{n} = n_x\vec{i} + n_y\vec{j} + n_z\vec{k}$ ; where  $n_x$ ,  $n_y$ , and  $n_z$  are the direction cosines of the unit normal.

Let  $\delta\vec{F}_n$  be the force that rest of the body is exerting on the plane  $ABC$ . Let  $\delta\vec{F}_{(-x)}$  be the force exerted by rest of the body on the plane  $AOB$  with unit normal  $-\vec{i}$ ; similarly we define force  $\delta\vec{F}_{(-y)}$ , for plane  $OBC$  with unit normals  $-\vec{j}$ ; and and force  $\delta\vec{F}_{(-z)}$  exerted on plane  $OAC$  with unit normal

---

\*©Rakesh K. Kapania, These notes are only for the use of students enrolled in AOE 3024, and those who have the author's permission.

$-\vec{k}$ . From Statics, we get;

$$\begin{aligned}\delta\vec{F}_n &= -(\delta\vec{F}_{(-x)} + \delta\vec{F}_{(-y)} + \delta\vec{F}_{(-z)}) \\ &= \delta\vec{F}_x + \delta\vec{F}_y + \delta\vec{F}_z\end{aligned}\quad (1)$$

Here we have made use of the fact that  $\delta\vec{F}_{(-x)} = -\delta\vec{F}_x$ ,  $\delta\vec{F}_{(-y)} = -\delta\vec{F}_y$ , and  $\delta\vec{F}_{(-z)} = -\delta\vec{F}_z$ .

Let  $\delta A$  be the area of plane  $ABC$ ; and  $\delta A_x$ ,  $\delta A_y$ , and  $\delta A_z$  that of planes  $OAB$ ,  $OBC$ , and  $OAC$ , respectively. The direction cosines of the unit normal  $\vec{n}$  can be expressed as:  $n_x = \delta A_x / \delta A$ ;  $n_y = \delta A_y / \delta A$ ; and  $n_z = \delta A_z / \delta A$ .

Equation (1), can be written in terms of stress components, as:

$$\vec{\sigma}_n \delta A = \vec{\sigma}_x \delta A_x + \vec{\sigma}_y \delta A_y + \vec{\sigma}_z \delta A_z \quad (2)$$

Dividing both sides by  $\delta A$ , the area of plane  $ABC$ , we get:

$$\begin{aligned}\vec{\sigma}_n &= \vec{\sigma}_x \frac{\delta A_x}{\delta A} + \vec{\sigma}_y \frac{\delta A_y}{\delta A} + \vec{\sigma}_z \frac{\delta A_z}{\delta A} \\ &= \vec{\sigma}_x n_x + \vec{\sigma}_y n_y + \vec{\sigma}_z n_z\end{aligned}\quad (3)$$

Note that (Eq. 3) is a vector equation, it thus represents a set of three equations, each one of which satisfies the equilibrium in  $x$ ,  $y$ , and  $z$  directions respectively. The three equations can be obtained by expressing both sides of Eq. (3) in terms of stress components. The stress vectors,  $\vec{\sigma}_n$ ,  $\vec{\sigma}_x$ ,  $\vec{\sigma}_y$ , and  $\vec{\sigma}_z$  in expanded form can be written as :

$$\begin{aligned}\vec{\sigma}_n &= \sigma_{nx}\vec{i} + \sigma_{ny}\vec{j} + \sigma_{nz}\vec{k} \\ \vec{\sigma}_x &= \sigma_{xx}\vec{i} + \tau_{xy}\vec{j} + \tau_{xz}\vec{k} \\ \vec{\sigma}_y &= \tau_{yx}\vec{i} + \sigma_{yy}\vec{j} + \tau_{yz}\vec{k} \\ \vec{\sigma}_z &= \tau_{zx}\vec{i} + \tau_{zy}\vec{j} + \sigma_{zz}\vec{k}\end{aligned}$$

Writing all the stress vectors, on both sides of Eq. 3, in expanded form, collecting the terms for each of the three unit base vectors  $\vec{i}$ ,  $\vec{j}$ , and  $\vec{k}$  on the right hand side; and comparing terms for each of the three unit base vectors, we get:

$$\begin{aligned}\sigma_{nx} &= \sigma_{xx}n_x + \tau_{yx}n_y + \tau_{zx}n_z \\ \sigma_{ny} &= \tau_{xy}n_x + \sigma_{yy}n_y + \tau_{zy}n_z \\ \sigma_{nz} &= \tau_{xz}n_x + \tau_{yz}n_y + \sigma_{zz}n_z\end{aligned}\quad (4)$$

The aforementioned set of equations constitute the well known (**Cauchy's**) relation for determining stress components on an arbitrary plane passing through the point for which the 9 stress components are known, provided the direction cosines of the unit normal of the plane are known.

Making use of the fact that the stress matrix is symmetric (we will prove it soon), Cauchy's relation, Eq. 4, can be written in the following matrix form:

$$\begin{Bmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = [\sigma] \{ \mathbf{n} \}$$

## 2 Lecture # 5, September 5 , 2001

The lecture was mainly about the use of Cauchy's relation to express boundary conditions and to calculate the Principal stresses and Principal planes. A *Principal plane* is a plane passing through the point of interest such that stress vector acting on the plane is normal to it. There are three such mutually orthogonal planes passing through the point. The corresponding value of the stress vector acting normal to each of the planes is called Principal Stress. The Principal planes are not subjected to any shear stresses.

**Boundary Conditions** Cauchy's relation is used for relating the surface tractions at a point on the surface (boundary) of the body to the inner stresses, or to determine the stress boundary conditions which must be satisfied at those points on the boundary where the tractions or surface forces are specified. Surface forces are the distributed forces acting on the surface and have dimensions of force per unit area. The fluid pressure acting on a body is an example of surface forces. Let the traction acting at a point on the boundary has components  $T_x^{(n)}$ ,  $T_y^{(n)}$  and  $T_z^{(n)}$ , and let  $n_x$ ,  $n_y$ , and  $n_z$  are the direction cosines of the outer unit normal acting on the surface at that point. Then the stress components at the boundary must be related to the three components of the surface traction as:

$$\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix}^{(n)} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

The above set of equations is same as the one given (without proof) as Eq. 1.7, page 9 of your text.

**Principal Stresses:** As mentioned before, at a given point the stress vector (its magnitude and direction) varies as the orientation of the plane, for which the stress vector is calculated, varies. A number of questions arise. Is there any plane(s) on which the normal stress has the largest possible value or what plane, passing through the point, has the largest possible shear stress value.

The knowledge of Principal planes answers these questions. A Principal plane is a plane such that the stress vector acting on that plane has no component which is tangent to the plane, *i.e.* there are no shear stresses acting on the plane. In other words, the stress vector has the same direction as the unit normal that describes the plane. The magnitude of the stress vector is called the Principal stress. Both the Principal plane and the corresponding stress can be obtained from the stress matrix as follows.

Let  $\bar{n}_x$ ,  $\bar{n}_y$ , and  $\bar{n}_z$  are direction cosines of the unit normal,  $\vec{n}$  of a Principal plane. Let  $\lambda$  is the magnitude of the stress vector acting on the plane. Since the stress vector and the unit normal are in the same direction, the stress vector  $\sigma_{\vec{n}}$  can be expressed as:

$$\sigma_{\vec{n}} = \lambda \vec{n} = \lambda(\bar{n}_x \vec{i} + \bar{n}_y \vec{j} + \bar{n}_z \vec{k})$$

From Cauchy's relation, the principal stress vector can be expressed in terms of the 9 components of the stress matrix as:

$$\begin{Bmatrix} \lambda \bar{n}_x \\ \lambda \bar{n}_y \\ \lambda \bar{n}_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \bar{n}_x \\ \bar{n}_y \\ \bar{n}_z \end{Bmatrix} \quad (5)$$

Equation 5 can be written as:

$$\begin{aligned} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \bar{n}_x \\ \bar{n}_y \\ \bar{n}_z \end{Bmatrix} &= \lambda \begin{Bmatrix} \bar{n}_x \\ \bar{n}_y \\ \bar{n}_z \end{Bmatrix} \\ &= \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{n}_x \\ \bar{n}_y \\ \bar{n}_z \end{Bmatrix} \end{aligned} \quad (6)$$

Equation (6) is an eigenvalue problem. By solving this eigenvalue problem, we can determine the value of  $\lambda$  and the direction cosines of the unit normal for the principal plane. In fact, as we will shortly see in the following, we have three (may or may not be distinct) possible values of  $\lambda$  and three corresponding planes.

Equation (6) can be written as:

$$\begin{bmatrix} \sigma_{xx} - \lambda & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - \lambda & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - \lambda \end{bmatrix} \begin{Bmatrix} \bar{n}_x \\ \bar{n}_y \\ \bar{n}_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (7)$$

One obvious solution is  $\bar{n}_x = 0$ ,  $\bar{n}_y = 0$ , and  $\bar{n}_z = 0$ , a trivial solution and of no particular value to us. From Kramer's rule, nontrivial solutions to Eq. (7) exist provided the determinant of the matrix on the left hand side of Eq. (7) vanishes. That is, to get nontrivial solutions, we must have:

$$\begin{vmatrix} \sigma_{xx} - \lambda & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - \lambda & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - \lambda \end{vmatrix} = 0 \quad (8)$$

Equation (8) is a cubic equation in  $\lambda$  and thus, has three solutions. These three solutions are called the eigenvalues of the stress matrix and give us three Principal stresses. Since the stress matrix is symmetric, we are guaranteed that the three eigenvalues (Principal stresses) will be real-valued.

Corresponding to each eigenvalue, we have a vector, called eigenvector, that yields the direction cosines of the unit normal of the corresponding Principal plane. The components of these vectors are obtained from Eq. (7) by substituting in this equation the value of  $\lambda$ , the principal stress of interest. Note that the three vector components of the eigenvector will not be uniquely

determined. This is because, once we substitute value of  $\lambda$  in Eq. (7), the set of the three equations is no longer a set of linearly independent equations. Any one of the three equations can be written as a linear combination of one or both of the remaining equations. To find nonunique solutions, we will have to assume a value for one (or two) of the three components and determine the remaining two (or one) in terms of assumed component(s).

The three components will thus only provide the overall direction of the normal to the Principal plane corresponding to a given  $\lambda$ . These components must be then normalized such that the magnitude of the eigenvector becomes unity.

These ideas are best understood using an example.

### 3 Lecture # 6, September 7 , 2001

Most of the time in this lecture was devoted to solving an example on Principal stresses and Principal planes.

**Example:** Determine the principal stresses and principal planes for the following state of stress.

$$[\sigma] = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using Eq. 7, the characteristic equation is given by the following equation:

$$\begin{vmatrix} 2 - \lambda & 2 & 0 \\ 2 & 2 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = 0$$

The above equation, on simplification, becomes:

$$(2 - \lambda) \begin{vmatrix} 2 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 \\ 0 & 1 - \lambda \end{vmatrix} + 0 \begin{vmatrix} 2 & 2 - \lambda \\ 0 & 0 \end{vmatrix} = 0$$

$$\text{or } (2 - \lambda)\{(2 - \lambda)(1 - \lambda) - 0\} - 2\{2(1 - \lambda) - 0\} = 0$$

$$\text{or } \underbrace{\lambda^3 - 5\lambda^2 + 4\lambda = 0}$$

Characteristic equation

The roots of the Characteristic equations are:  $\lambda_1 = 4$ ,  $\lambda_2 = 1$ , and  $\lambda_3 = 0$ . These are the three principal stress values.

**Principal Planes** The unit vectors for each of the three principal planea can be obtained by substituting the values of each of the three principal stresses in Eq. (7). Let  $\vec{\mathbf{n}}^{(l)}$  be the eigenvector corresponding to the  $l$ th eigenvalue, where  $l = 1, 2, 3$ .

**Case l=1:** The eigenvector for the the first eigenvalue  $\lambda_1 = 4$  can be obtained by substituting  $\lambda = 4$  in Eq. 7. This leads to a set of following three equations:

$$\begin{bmatrix} 2-4 & 2 & 0 \\ 2 & 2-4 & 0 \\ 0 & 0 & 1-4 \end{bmatrix} \begin{Bmatrix} \bar{n}_x \\ \bar{n}_y \\ \bar{n}_z \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (9)$$

The first two linear equations in the set of equations given by (9) are exactly the same. Thus the three unknowns are not independent of each other. We need to describe two of the unknowns in terms of the third one, say  $\bar{n}_1^{(1)}$ . A solution to the set of equations given by (9) is:

$$\bar{n}_1^{(1)} = 1, \quad \bar{n}_2^{(1)} = 1, \quad \text{and} \quad \bar{n}_3^{(1)} = 0$$

The above values can be normalised to yield the magnitude of the vector  $\vec{\mathbf{n}}^{(1)}$  to be 1. The normalized eigenvector or the unit normal describing the Principal plane corresponding to the first Principal stress is given as:

$$\vec{\mathbf{n}}^{(1)} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$$

**Case l=2, and l=3:** The normalised eigenvectors corresponding to the second and third eigenvalues (Principal stresses) can be obtained following a similar procedure. These are, respectively, for  $\lambda_2 = 1$ , and  $\lambda_3 = 0$ ,

$$\vec{\mathbf{n}}^{(2)} = \vec{k}$$

$$\vec{\mathbf{n}}^{(3)} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$$

The fact that one of the eigenvector is  $\vec{k}$  should not be a surprise as the given state of stress has no shear components on the  $z$ - plane. The  $z$ -

plane is thus one of the Principal planes with the Principal stress being 1. The eigenvalue analysis merely verifies that fact.

Also note that the three Principal planes given, respectively, by three eigenvectors  $\vec{\mathbf{n}}^{(1)}$ ,  $\vec{\mathbf{n}}^{(2)}$ , and  $\vec{\mathbf{n}}^{(3)}$  are mutually orthogonal.

To determine failure, at a point, it is often of interest to find **maximum shear stress** and corresponding plane. Using Mohr's circle, the maximum shear stress, can be obtained as:

$$\tau_{max} = \frac{\lambda_1 - \lambda_3}{2}$$

Here  $\lambda_1$  and  $\lambda_3$  are the largest and the smallest (in algebraic sense, and not in terms of absolute value) principal stresses. This stress will act on the plane that makes an angle of  $\pi/4$  with both  $\vec{\mathbf{n}}^{(2)}$ , and  $\vec{\mathbf{n}}^{(3)}$  planes.