AOE 3024: Thin Walled Structures
Mohr's Circle For Plane Stress

THEORY

The three dimensional state of stress for a plane stress problem reduces to three independent components,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

The plane stress transformation formulas are

$$\overline{\sigma}_{xx} = \sigma_{ave} + \sigma_{dif}\,\cos 2\theta + \tau_{xy}\,\sin 2\theta \tag{2a}$$

$$\overline{\sigma}_{yy} = \sigma_{ave} - \sigma_{dif}\,\cos 2\theta - \tau_{xy}\,\sin 2\theta \tag{2b}$$

$$\overline{\tau}_{xy} = -\sigma_{dif}\,\sin 2\theta + \tau_{xy}\,\cos 2\theta \tag{2c}$$

where

$$\sigma_{ave} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \qquad \sigma_{dif} = \frac{\sigma_{xx}-\sigma_{yy}}{2}$$

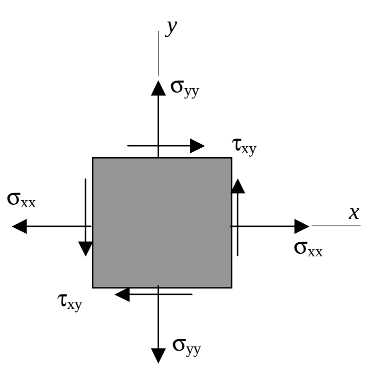
The transformation equations for plane stress can be represented in graphical form by a plot known as Mohr's circle. A more rigorous derivation can be found in the course textbook and references.

Step I. Calculate the radius and Mohr's circle center

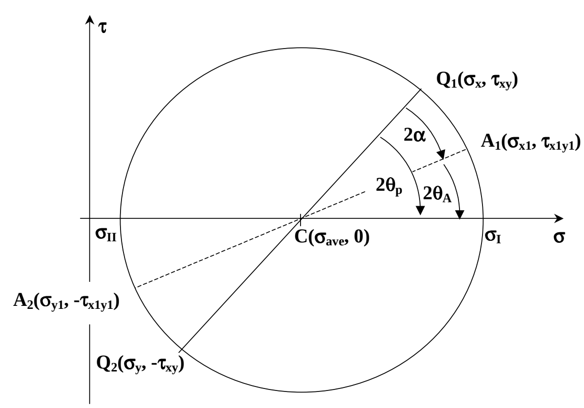
$$\sigma_{ave} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \qquad \sigma_{dif} = \frac{\sigma_{xx}-\sigma_{yy}}{2} \tag{3}$$

$$R = \sqrt{\tau_{xy}^2 + \sigma_{dif}^2} \qquad C = C(\sigma_{ave}, 0) \tag{4}$$

Step II. Draw the circle and locate all points



a) Positive stresses on a two dimensional element



b) Mohr's circle for plane stress in the xy plane. Point **Q** is the location of the defined state of stress. Point **A** is the location in the Mohr's circle where information is desired.

Fig. 1

Step III. Calculate angles: (All measured positive clockwise)

Principal stresses act on an element inclined at an angle θ_p

$$\tan 2\theta_p = \frac{\tau_{xy}}{\sigma_{dif}} \tag{5a}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left[\frac{\tau_{xy}}{\sigma_{dif}}\right] \tag{5b}$$

Maximum shear stresses act on an element inclined at an angle θ_s

$$2\theta_s = 2\theta_p \pm 90^\circ \tag{6a}$$

$$\theta_s = \theta_p \pm 45^\circ \tag{6b}$$

Transformed stresses act on an element inclined at an angle α

$$2\,\theta_A = 2\,\theta_p - 2\,\alpha \tag{7a}$$

Note that all angles are measured positive clockwise in the Mohr's circle but are positive counterclockwise in the rotation of the differential element. Also, note that $2\,\theta_p$ is measured from $\overline{Q_1C}$ to positive σ-axis.

Step IV. Determine the normal stresses

The normal stresses acting on an element inclined at an angle α are

$$\sigma_{x_1} = \sigma_{ave} + R\cos(2\,\theta_A) \tag{8}$$

$$\sigma_{y_1} = \sigma_{ave} + R\cos(2\,\theta_A + 180^\circ) \tag{9}$$

$$= \sigma_{ave} - R\cos(2\,\theta_A) \tag{10}$$

Note that when calculating principal stresses $2\,\alpha = 2\,\theta_p \rightarrow 2\,\theta_A = 0^\circ$, therefore the principal stresses are

$$\sigma_I = \sigma_{ave} + R \tag{11}$$

$$\sigma_{II} = \sigma_{ave} - R \tag{12}$$

Step V. Determine the shear stress

The shear stresses acting on an element inclined at an angle α are

$$\tau_{x_1y_1} = R\sin(2\,\theta_A) \tag{13}$$

The maximum shear stresses acting on an element inclined at an angle θ_s are

$$\tau_{max}\Big|_{in-plane} = R = \frac{\sigma_I - \sigma_{II}}{2} \tag{14}$$

Step VI. Show all results on sketches of properly oriented elements

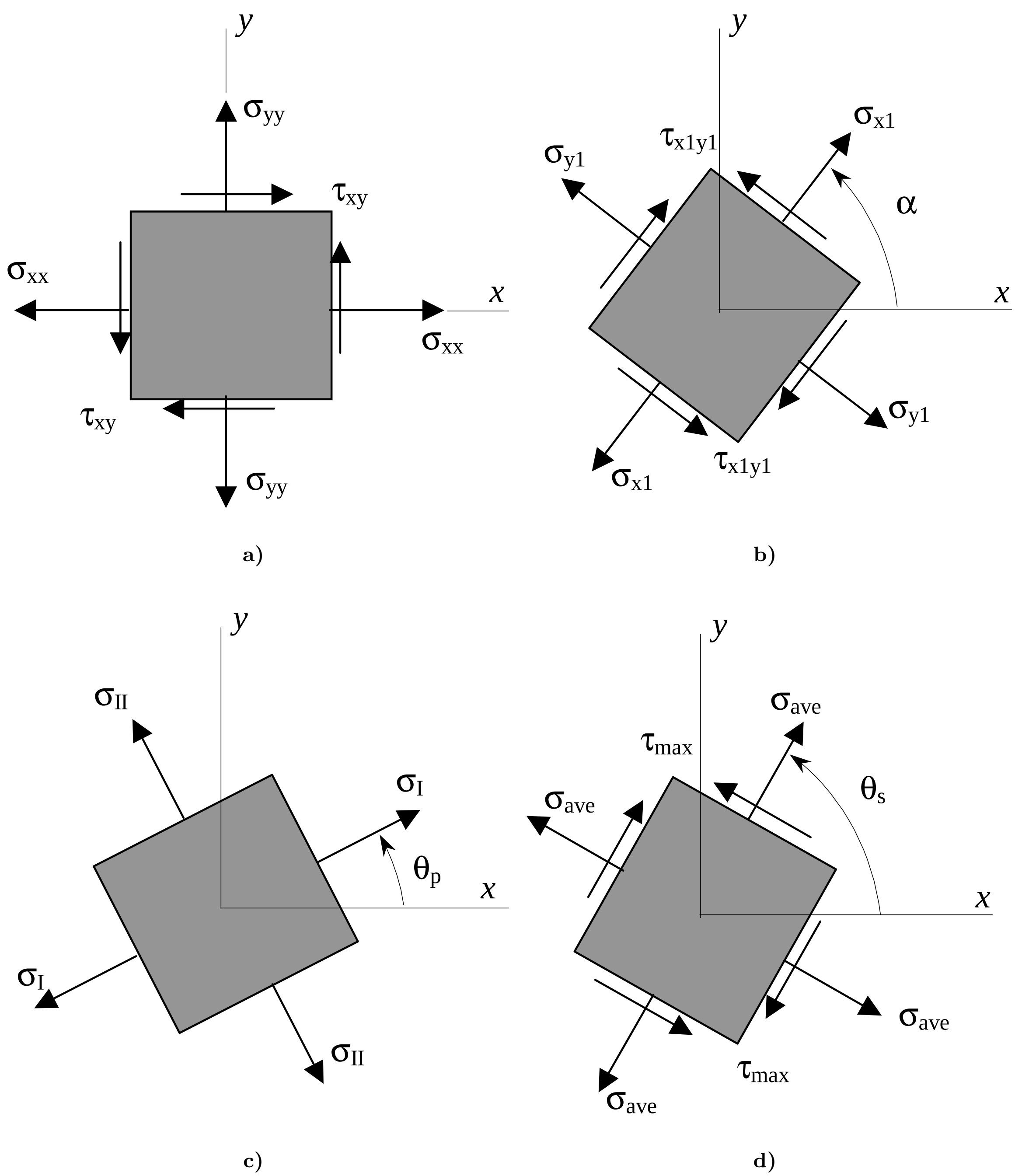


Fig. 2 a) Stresses acting on an element in plane stress. b) Stresses acting on an element oriented at an angle $\theta = \alpha$. c) Principal normal stresses. d) Maximum shear stresses.

EXAMPLE # 1

At a point on the surface of a generator shaft the stresses are

$$\boldsymbol{\sigma} = \begin{bmatrix} -50 & -40 & 0 \\ -40 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ \textbf{MPa} \tag{15}$$

Using Mohr's circle and only considering in-plane stresses, determine the following quantities: a)stresses acting on an element inclined at an angle $\alpha = 40°$; b) principal stresses; c) maximum shear stresses

Step I. Calculate the radius and Mohr's circle center

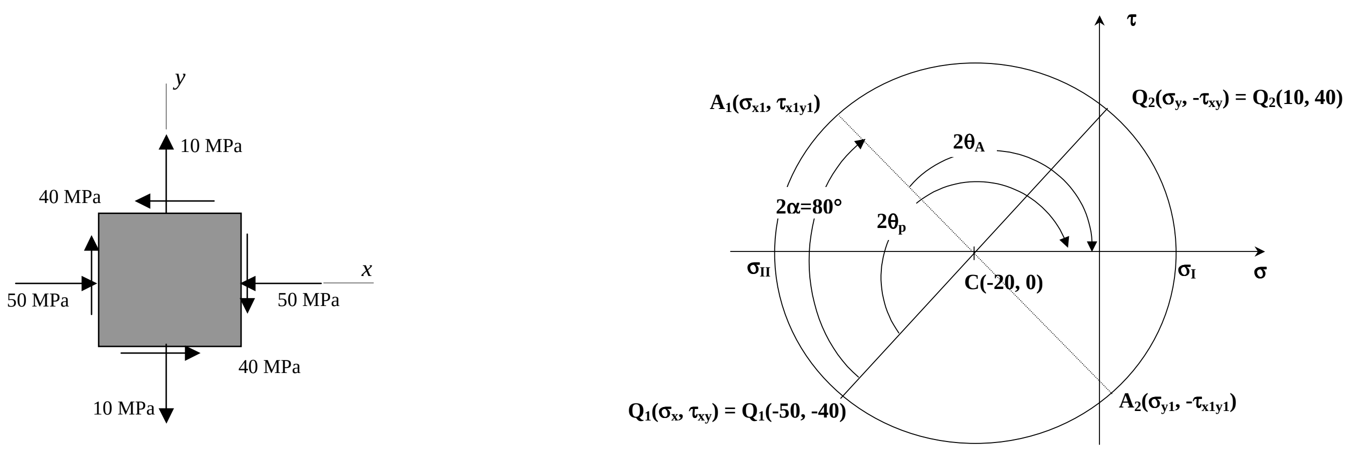
$$\sigma_{ave} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{(-50) + (10)}{2} \text{ MPa} = -20 \text{ MPa} \tag{16}$$

$$\sigma_{dif} = \frac{\sigma_{xx} - \sigma_{yy}}{2} = \frac{(-50) - (10)}{2} \text{ MPa} = -30 \text{ MPa} \tag{17}$$

$$R = \sqrt{\tau_{xy}^2 + \sigma_{dif}^2} = \sqrt{(-40)^2 + (-30)^2} \text{ MPa} = 50 \text{ MPa} \tag{18}$$

$$C = C(\sigma_{ave}, 0) = C(-20 \text{ MPa}, 0) \tag{19}$$

Step II. Draw the circle and locate all points



a) stresses on a two dimensional element

b) Mohr's circle for plane stress in the xy plane

Fig. 3

Step III. Calculate angles: (All measured positive clockwise)

Principal stresses act on an element inclined at an angle θ_p

$$2\theta'_p = \tan^{-1}\left[\frac{\tau_{xy}}{\sigma_{dif}}\right] = \tan^{-1}\left[\frac{(-40)}{(-30)}\right] = 53.135° \tag{20a}$$

$$2\theta_p = 2\theta'_p + 180° = 233.13° \tag{20b}$$

$$\theta_p = 116.565° \tag{20c}$$

Note that we add 180° because $2\,\theta_p$ is measured from $\overline{Q_1 C}$ to positive σ-axis. Maximum shear stresses act on an element inclined at an angle θ_s

$$2\theta_s = 2\theta_p \pm 90^\circ = 233.13^\circ \pm 90^\circ \tag{21a}$$

$$\theta_s = \theta_p \pm 45^\circ = 116.565^\circ \pm 45^\circ \tag{21b}$$

Transformed stresses act on an element inclined at an angle $\alpha = 45^\circ$

$$2\,\theta_A = 2\,\theta_p - 2\,\alpha = 233.13^\circ - 80^\circ = 153.13^\circ \tag{22a}$$

Note that all angles are measured positive clockwise in the Mohr's circle but are positive counterclockwise in the rotation of the differential element.

Step IV. Determine the normal stresses

The normal stresses acting on an element inclined at an angle α are

$$\sigma_{x_1} = \sigma_{ave} + R\,\cos(2\,\theta_A) = (-20) + (50)\,\cos(153.13^\circ) = -64.6018 \text{ MPa} \tag{23}$$

$$\sigma_{y_1} = \sigma_{ave} - R\,\cos(2\,\theta_A) = (-20) - (50)\,\cos(153.13^\circ) = 24.6018 \text{ MPa} \tag{24}$$

Note that when calculating principal stresses $2\,\alpha = 2\,\theta_p \rightarrow 2\,\theta_A = 0^\circ$, therefore the principal stresses are

$$\sigma_I = \sigma_{ave} + R = (-20) + (50) = 30 \text{ MPa} \tag{25}$$

$$\sigma_{II} = \sigma_{ave} - R = (-20) - (50) = -70 \text{ MPa} \tag{26}$$

Step V. Determine the shear stress

The shear stresses acting on an element inclined at an angle α are

$$\tau_{x_1 y_1} = R\,\sin(2\,\theta_A) = (50)\,\sin(153.13^\circ) = 22.5983 \text{ MPa} \tag{27}$$

The maximum shear stresses acting on an element inclined at an angle θ_s are

$$\tau_{max}\Big|_{in-plane} = R = \frac{\sigma_I - \sigma_{II}}{2} = 50 \text{ MPa} \tag{28}$$

Step VI. Show all results on sketches of properly oriented elements

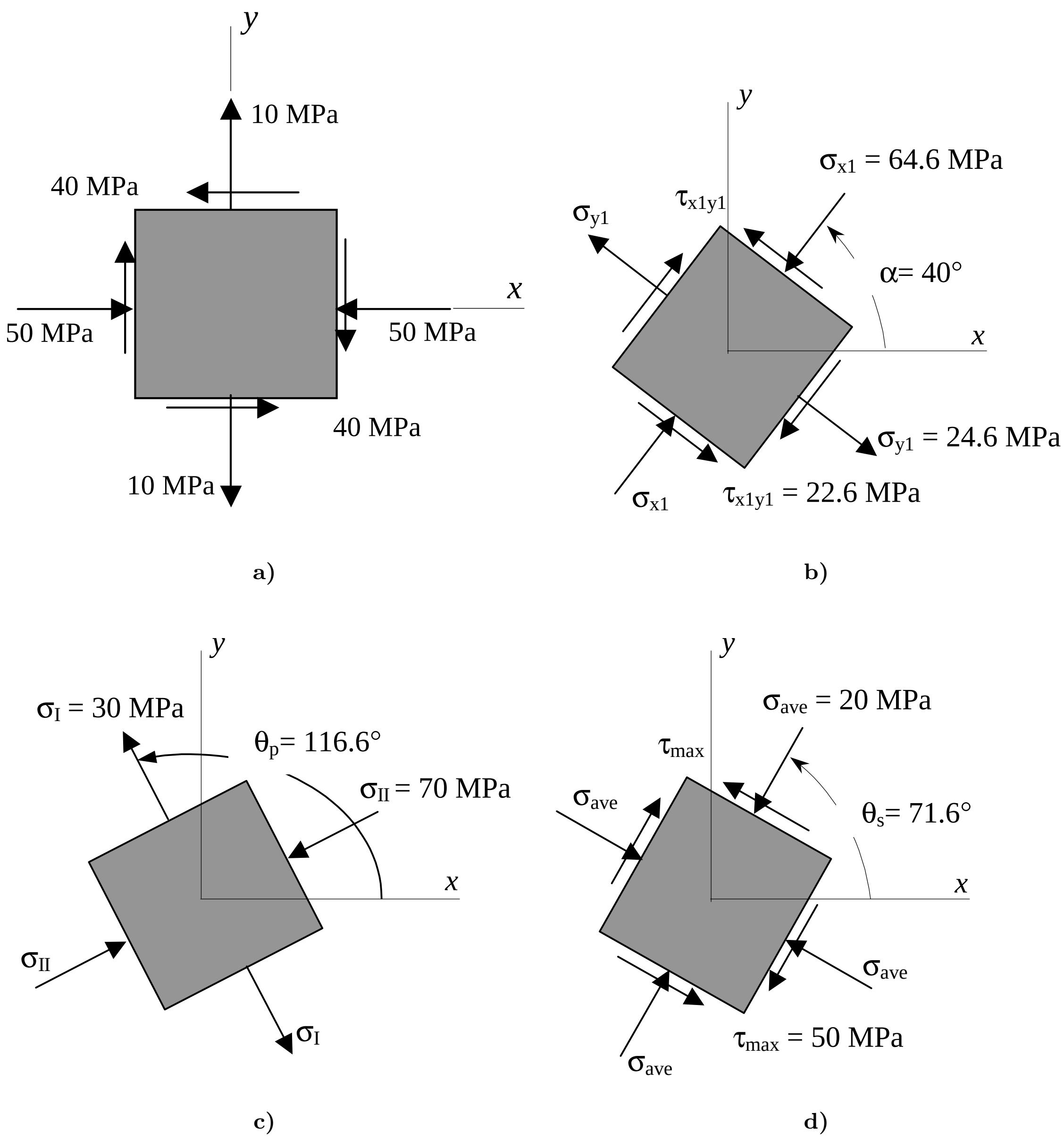


Fig. 4 a) Stresses acting on an element in plane stress. b) Stresses acting on an element oriented at an angle $\theta = \alpha$. c) Principal normal stresses. d) Maximum shear stresses.