Review Notes for Week 8

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October 29, 2001

Recall that during the previous week, Week#7, we mainly concentrated on two topics (i) Understanding Shear force and bending Moment diagram (ii) Review of such concepts as the section centroid, section second moments of area, and the application of the parallel axis theorem. Given the second moment of an area about its own centroidal axes, the second moment of the area about another set of axes can be obtained using the parallel axis theorem. Since these topics are really review of the material covered in mechanics of materials, the book by Beer and Johnston is an excellent source for the material covered in Week #7.

During the Week #8, we mainly focused on determining the bending stresses in a beam with unsymmetric cross-section.

For the analysis, the beam is considered to be placed along the $z-$axis and the cross-section is in the x-y plane. A beam is considered to have unsymmetric cross-section if the cross-section does not have at least one axis of symmetry. For such a cross-section, the product moment of area, I_{xy}, does not vanish. As a result, when subjected to a bending moment, there exists a coupling between the resulting curvatures in the $x - z$ and $y - z$ planes.

This means that a transverse load in, say $x - z$ plane will cause a transverse deflection in both $x - z$ and $y - z$ planes. If the beam has a symmetric cross-section (*i.e.* $I_{xy} = 0$), this does not happen. For such a beam, the deflection of the beam is in the same plane as the applied transverse load.

The material essentially follows that given in Chapter 9 of your book.

1 Lecture # 1, October 15, 2001

TOne of the objectives of this lecture was to explain the sign convention used in the book. The key point is that the bending moments M_x, the bending moment about the $x-$axis, and M_y, the bending moment about the $y-$axis are considered positive if they cause tensile stresses in the first quadrant (*i.e.*both x and y are positive). Note that the bending moment M_x is the bending moment caused by the transverse loads applied in the $y - z$ plane (for example, a distributed force w_y in the y-direction) and M_y is the bending moment caused by the applied transverse loads in the $x - z$ plane (for example, a distributed force w_x in the x-direction).

Note that if we represent the bending moment as a 'vector' with two arrows, M_x will be positive if the vector that represents the moment is in the *positive* $x-$direction, but M_y is positive if the vector that represents the moment is in the *negative* $y-$direction. Recall that in the 'vector' representation of a moment, if one looks in the direction of the double-arrow vector, the moment is in the clock-wise direction.

We also reviewed the idea of curvature and radius of curvature for a curve. When a beam bends in the form of a curve, the notion of curvature (more precisely, its inverse, the radius of curvature) helps us to determine the strain in any fiber in the beam. To understand curvature of a curve, let us consider v, the displacement in the $y-$ direction as a function of z, the axis along the length of the beam. Curvature κ_x at a point P of a curve in the $v - z$ plane with s, the length of the arc measured along the curve, as a parameter along the curve is defined as:

$$\kappa_x = \frac{1}{\rho_x} = \frac{d\theta_x}{ds} \tag{1}$$

Here θ_x is the angle that the tangent to the curve at point P makes with the z-axis (see Fig. 1); and ρ_x is the radius of curvature at P. Consider

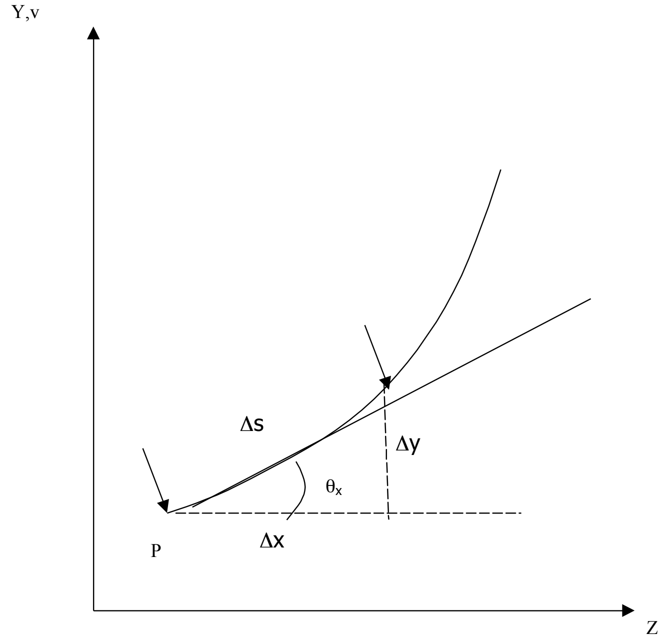

Figure 1: Illustration of various quantitities in defining curvature $\kappa = d\theta_x/ds$

another point Q in the neighbourhood of P at a distance Δs. As the point Q approaches P, we can write:

$$\cos\theta_x = \lim_{\Delta s \to 0} \frac{\Delta z}{\Delta s} = \frac{dz}{ds} = \left(\frac{1}{1 + (dv/dz)^2} \right)^{1/2}$$

$$\sin\theta_x = \lim_{\Delta s \to 0} \frac{\Delta v}{\Delta s} = \frac{dv}{ds} = \frac{dv}{dz} \left(\frac{1}{1 + (dv/dz)^2} \right)^{1/2}$$

$$\tan\theta_x = \frac{dv}{dz}$$

Here, we have made use of the fact that $ds = \sqrt{dv^2 + dz^2} = dz\sqrt{1 + (dv/dz)^2}$. Also, dv/dz represents the slope of the curve at P. In our analysis of beams, we assume that the rotation of the beam after bending is very small compared to unity, *i.e.* $dv/dz << 1$.

Taking the derivative of the expression for $\tan\theta_x$ with respect to ds, we can write:

$$\begin{aligned} \frac{d(\tan\theta_x)}{ds} &= \frac{d}{ds}\left(\frac{dv}{dz}\right) = \frac{dz}{ds}\frac{d}{dz}\left(\frac{dv}{dz}\right) = \cos\theta_x \left(\frac{d^2v}{dz^2}\right) \\ \sec^2\theta_x \frac{d\theta_x}{ds} &= \cos\theta_x \left(\frac{d^2v}{dz^2}\right) \end{aligned} \qquad (2)$$

From the above equation, we get:

$$\begin{aligned} \frac{d\theta_x}{ds} &= \cos^3\theta_x \frac{d^2v}{dz^2} \\ &= \frac{d^2v}{dz^2}\left(\frac{1}{1+(dv/dz)^2}\right)^{3/2} \\ &\approx \frac{d^2v}{dz^2} \quad \text{for} \quad dv/dz << 1 \end{aligned} \tag{3}$$

The radius of curvature, thus, *for small beam rotations* is given as:

$$\kappa_x = \frac{1}{\rho_x} \approx \frac{d^2v}{dz^2} \tag{4}$$

Similarly, for the deflection of the beam u, in the x-z plane, the radius of curvature ρ_y can be written as:

$$\kappa_y = \frac{1}{\rho_y} \approx \frac{d^2u}{dz^2} \tag{5}$$

We reiterate that the above expression for the curvature of the beam after deformation assumes that the beam bends with a small rotation, i.e. $du/dz << 1$.

2 Lecture # 2, October 17,2001

Recall that the key objective here is to determine the stresses at any point in a beam under the action of a bending moment applied loads. Important steps in achieving this goal are :(i) determining the strain at any point in terms of beam displacements (strain-displacement relations); (ii) Relating these strains to stresses using stress-strain relation (Hooke's law); and (iii) relating the stresses to the applied bending moment. Obviously, it is being assumed that given a transverse load distribution, the student is able to obtain the bending moment at any section of the beam by drawing the bending moment diagram.

Strictly speaking, any structure under an arbitrary loads should have a three dimensional state of stress. That means, at each point, one must determine all the six stress components. However, the fact that the beam is very long in one dimension when compared to the other two dimensions allows us to make certain asuumptions about the state of stress in a beam that greatly simplifies the analysis. In fact, these assumptions aimplify the analysis so much so that, for a large number of beams, one only has to worry about just one stress component, namely σ_z, the normal stress in the direction of z, the beam axis. The assumptions, which allow us to achieve this level of simplicity, are the so-called **Euler-Bernoulli** assumptions. These are:

1. Plane sections before deformation remain plane after deformation. This assumption ensures that the axial strain ϵ_z is a linear function of the cross-sectional co-ordinates x and y.

2. Two fibers mutually orthogonal before deformation remain mutually orthogonal after deformation. This assumption assures that there are no shear strains, *i.e.* $\gamma_{xz} = \gamma_{yz} = 0$. If the material is isotropic, as is assumed here, $\tau_{xz} = \tau_{yz} = 0$

3. The material is linearly elastic with Young's modulus E.

4. The normal stresses σ_x and σ_y and the shear stress τ_{xy} are negligible.

Note that these are only assumptions and they may or may not be true for all possible cases. But if you have a very long (beam length typically greater than 15-20 times the largest cross-sectional dimension) isotropic beam, you can use these assumptions without committing a significant error.

Based on the first two assumptions, the deformed shape, $A'B'C'D'$, in the y-z plane, of a beam differential element $ABCD$, of length Δz, is shown in Fig. 2. Note that the fiber towards the top of the element are under compression and those towards the bottom of the differential element are under tension. That means, there is a fiber in the differential element which stays unstrained, say the fiber PQ. The complete deformation of the beam can then be expressed by the displacement v of this unstretched fiber (called neutral axis) in the y-direction. Note that the transverse deflection v is positive if it is in the positive y direction. As the length of the differential element approaches zero, the deformed shape of the beam approaches a circle of radius ρ_x, see Eq. 4, given as

$$\rho_x \approx \frac{d^2 v}{dz^2}$$

The axial strain in any fiber, say RS before deformation and $R'S'$ after deformation, lying at a distance y from the neutral axis, can be written as:

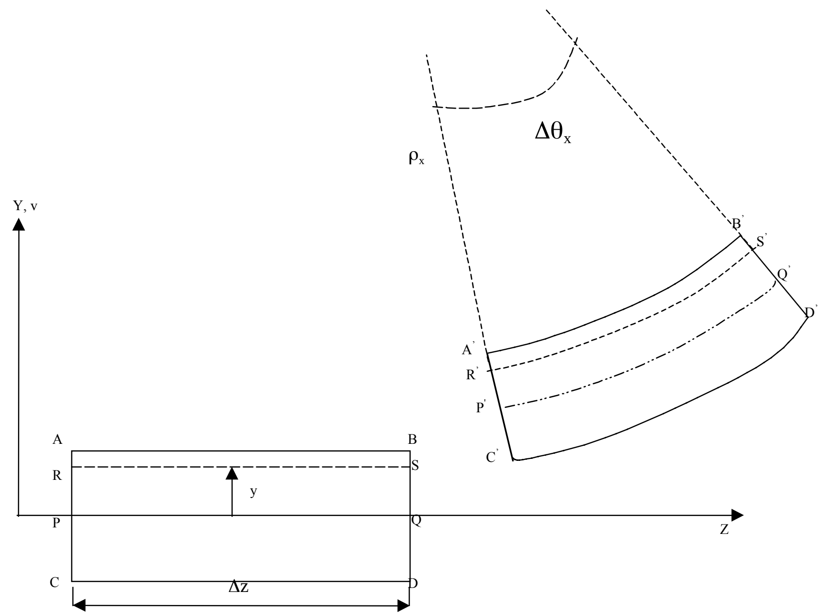

Figure 2: Bending of a beam element of length dz into a circular arc of radius ρ_x in $y - z$ plane.

$$\begin{aligned}\epsilon_z(y) &= \frac{R'S' - RS}{RS} \\ &= \frac{R'S' - PQ}{PQ} = \frac{R'S' - P'Q'}{P'Q'} \\ &= \frac{(\rho_x - y)\Delta\theta_x - \rho_x\Delta\theta_x}{\rho_x\Delta\theta_x} \\ &= -\frac{y}{\rho_x} \approx -y\frac{d^2v}{dz^2} \qquad (6)\end{aligned}$$

Note that the axial strain ϵ_z due to the displacement in the y-z plane is a linear function of y.

Using similar arguments, the axial strain ϵ_z, for a fiber at a distance x from the unstretched fiber, can be written as:

$$\epsilon_z(x) = -\frac{x}{\rho_y} \approx -x\frac{d^2u}{dz^2} \qquad (7)$$

Here u is the displacement of the unstretched (thus unstressed) fiber in the x direction, and is considered positive if it is in the positive x-direction.

Combining Eqs. 6 and 7, the total axial strain ϵ_z at a point x, y, measured from the unstretched fiber (neutral axis), can be expressed as:

$$\epsilon_z(x,y) = -\frac{x}{\rho_y} - \frac{y}{\rho_x} \approx -x\frac{d^2u}{dz^2} - y\frac{d^2v}{dz^2} \tag{8}$$

We repeat that the distances x and y are measured with respect to a point such that the axial fiber passing through that point remains unstressed. In the subsequent lecture, it will be shown that for a homogeneous section, the point is the centroid of the cross-section.

The bending stress σ_z hence can be obtained using the constitutive behavior of the material (*i.e.* using the Hooke's law):

$$\sigma_z = -E\left(\frac{x}{\rho_y} + \frac{y}{\rho_x}\right) \approx -E\left(x\frac{d^2u}{dz^2} + y\frac{d^2v}{dz^2}\right) \tag{9}$$

To determine the σ_z using Eq. 9, we need the values of both ρ_x, and ρ_y in terms of the applied moments M_x and M_y. Also, we need to determine the location of the point through which the fiber remains unstretched, *i.e.* the neutral axis, since both x and y are measured from this point. These objectives are met by using the fact that the bending stress σ_z must be statically equivalent to the applied bending moments M_x and M_y and applied axial force (which of-course is zero).

3 Lecture # 3, October 19, 2001

This lecture was mostly devoted to proving that even for the case of a beam with unsymmetric cross-section, the neutral axis passes through the centroid. In essence, what it means is the co-ordinates x and y in Eq. 8 is measured from the section **centroid**. The proof follows the one given in Chapter 9 of your book. This is proved satisfying the requirement that the net axial force created by the the stress σ_z should be such that it is statically equivalent to the applied force F_z in the z-direction. Since the only external force which we are considering are the bending moments M_x and M_y, the external axial force causing the bending stress is zero. This implies that the net axial force resultant of σ_z should be zero.

The net axial force resultant can be obtained by integrating the net axial force,$dF_z = \sigma_z dA$ on a differential are dA due to σ_z over the entire area of cross-section. The total axial force is given as:

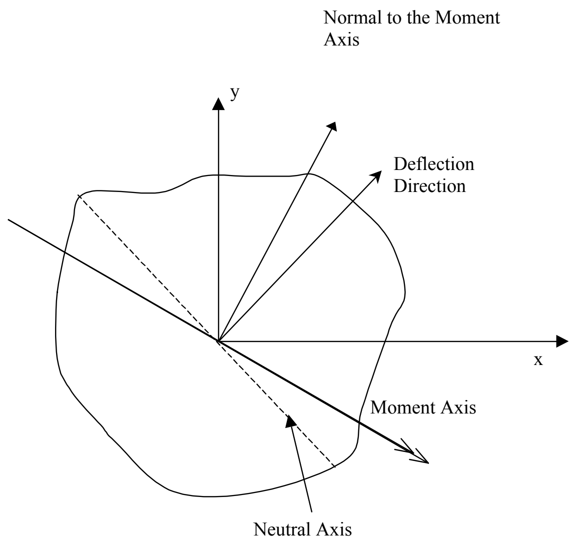

Figure 3: Representation of the moment axis and the neutral axis for an unsymmetric section. Note that the two axes coincide for a beam section with $I_{xy} = 0$, but not in general if $I_{xy} \neq 0$

$$F_z = \iint_{\text{Area}} \sigma_z dA = 0$$

For details of the proof to show that the neutral axis passes through the centroid, please see p 279 of your text.

Note that for an unsymmetric cross-section, the beam does not bend in the same plane in which the resultant transverse load is applied. The zero bending stress line in the beam cross-section, which is always normal to the transverse deflection of the beam, thus, is not normal to the resultant of the applied transverse load as is the case for a beam with a cross-section for which the product of inertia $I_{xy} = 0$. Alternately, the zero stress line in the cross-section does not coincide with the moment axis, the axis about which the resultant moment acts. This is explained in Fig. 3 where we indicate, on a beam crosse-section, the zero stress line, the resultant moment axis, and the direction of the transverse deflection.

For the sake of completeness, we mention that, it is possible to pick a co-ordinate system, say x'-y', in the cross-sectional plane, such that the product section moment of inertia vanishes, *i.e.* $I_{x'y'} = 0$. Such a co-ordinate system is called principal co-ordinate system and can be determined using Mohr circle. Once this new co-ordinate system has been determined, the beam bending problem can be solved by using the relations available for symmetric bending. Of course, you will need to first transform the applied loads, known in the x and y directions, into those in x' and y' directions. This is accomplished using a standard co-ordinate transformation of components of a vector. Also, you will need to find $I_{x'x'}$ and $I_{y'y'}$ in terms of I_{xx}, I_{yy}, and I_{xy}. Because of all these complexities, often little is gained by using this approach of employing the principal co-ordinate system.