AOE 3024: Thin Walled Structures
Solutions to Homework # 10

Problem 9.15: For the unsymmetrically thin-walled section shown, determine the location of the shear center.

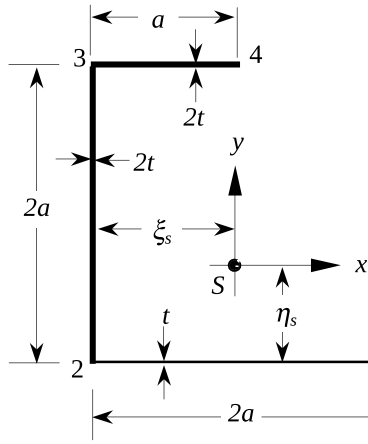

Fig. 1 Unsymmetrically thin-walled channel section.

FIRST: Understand the problem

One can often reduce the amount of computation by giving some thought to the problem:

Since this is a unsymmetrically thin-walled section, we will need to calculate all second moments of area about the centroid. Therefore, the location of the centroid will be needed.

Next, before we proceed let us decide where to take the torque equivalence. Recall that the torque equivalence about a point, say O_s, determines the location of the shear center. Further, arbitrarily apply shear loads S_y and S_x at the shear center S.

Torque equivalence can be taken about any point. However, for some points less calculation will be needed. For instance, if we take moment about the shear center then the shear flow acting at each branch will have to be calculated. If we choose point 1 or 4, then the shear flow acting in two branches will have to be calculated.

For point 2 or 3, only the shear flow along one of the branches needs to be calculated. If torque equivalence is taken about a point 3, then only calculate $q_{12}(s)$. If torque equivalence is taken about a point 2, then only calculate $q_{43}(s)$. Therefore, for the solutions to this problem point 3 is used.

Regardless about what point we take the torque equivalence, we should get the same answer.

If point $O_s = 3$, then we should consider Fig. 2

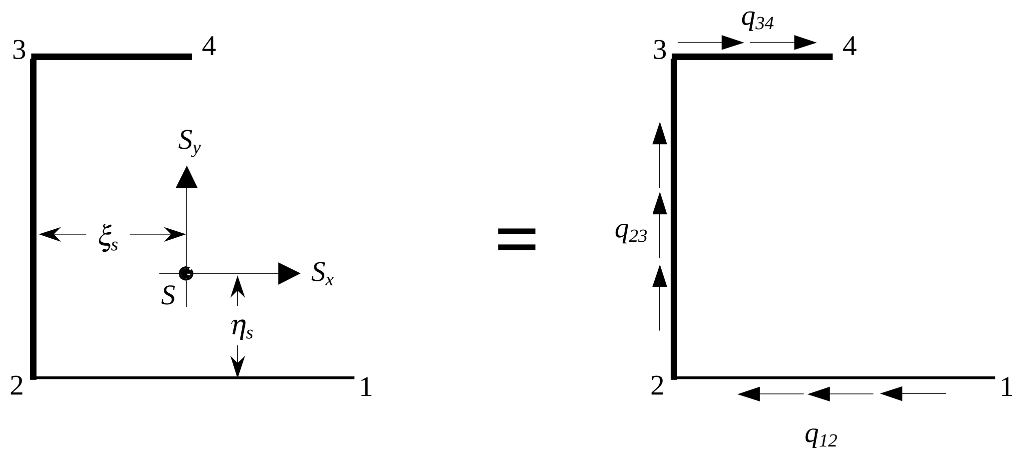

Fig. 2 Suggested shear flow convention for statically equivalence by taking the torque at point 3

If point $O_s = 2$, then we should consider Fig. 3

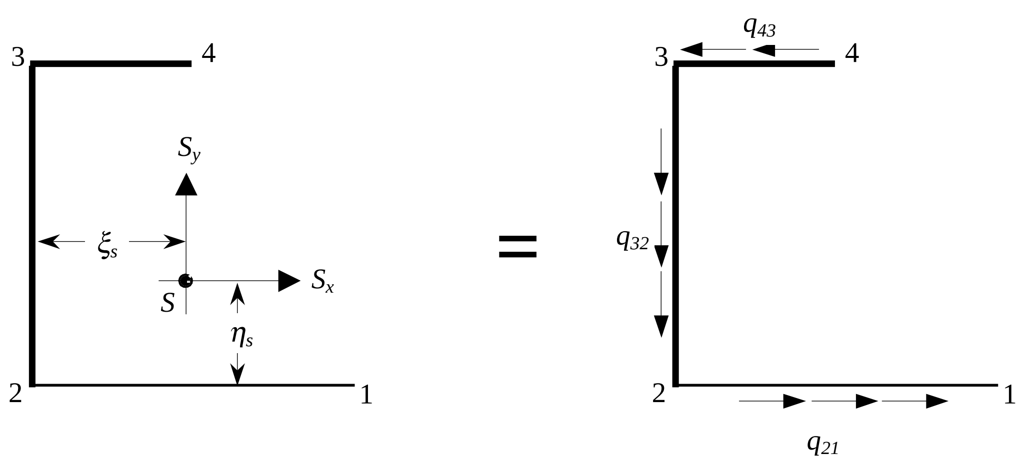

Fig. 3 Suggested shear flow convention for statically equivalence by taking the torque at point 2

It is not necessary to follow the above shear convention. However, in doing so it simplifies the computations. As a practice, we will here calculated the shear flow distribution for all three branches. Further note that:

$$q_{ij}(s) = -q_{ji}(s)$$

For the present solution we will take torque equivalence about point 3 and use Fig. 2. In solving this problem, consider a four-step solution procedure:

1. Calculate the centroid: x_c and y_c
2. Determine the second moments of area about the centroid: I_{xx}, I_{yy} and I_{xy}
3. Find the shear flow distribution in each branch, $q_i(s)$, to find the vertical location, η_s, of the shear center by considering $S_y = 0$ and $S_x > 0$
4. Find the shear flow distribution in each branch, $q_i(s)$, to find the horizontal location, ξ_s, of the shear center by considering $S_x = 0$ and $S_y > 0$

BASIC EQUATIONS:

Shear flow distribution

When S_x and S_y is applied through the shear center then no torsion exists and the shear flow distribution is given by

$$q_i(s) = q_{o_i} - \left(\frac{S_x\, I_{xx} - S_y\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s t_i(s_i)\, x_i(s_i)\, ds_i - \left(\frac{S_y\, I_{yy} - S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s t_i(s_i)\, y_i(s_i)\, ds_i \qquad (1)$$

where q_{o_i} is found by evaluating $q_i(s=0)$.
When the thickness is constant along each branch: $t_i(s_i) = t_i$.

Torque equivalence

Torque equivalence about a point, say $O_s = 3$, determines the location ξ_s of the shear center. For a positive torque counterclockwise (see Fig. 2),
for $S_y = 0$ and $S_x > 0$

$$S_x\,(2a - \eta_s) = \sum_i \int_0^{a_i} \overrightarrow{r}_i(s) \times \overrightarrow{q}_i(s)\, ds \qquad (2)$$

for $S_x = 0$ and $S_y > 0$

$$S_y\, \xi_s = \sum_i \int_0^{a_i} \overrightarrow{r}_i(s) \times \overrightarrow{q}_i(s)\, ds \qquad (3)$$

where i represents the i^{th} branch, a_i the length of the branch, and $r_i(s)$ the distance perpendicular from the point 3 to the contour s_i.

I. Centroid

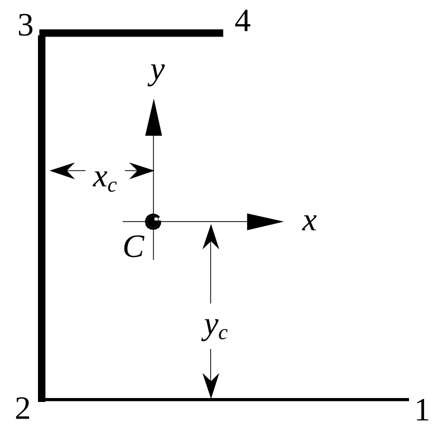

Fig. 4 Calculation of the centroid.

Let's place the origin at point 2 and calculate x_c and y_c.

Section	x_i	y_i	A_i
12	a	0	$(2a)(t)$
23	0	a	$(2t)(2a)$
34	$a/2$	$2a$	$(a)(2t)$

$$\begin{aligned} \sum A_i &= 8\,a\,t \\ \sum x_i\,A_i &= 3\,a^2\,t \\ \sum y_i\,A_i &= 8\,a^2\,t \end{aligned}$$

$$\begin{aligned} x_c &= \frac{\sum x_i\,A_i}{\sum A_i} = \frac{3}{8}\,a \\ y_c &= \frac{\sum y_i\,A_i}{\sum A_i} = a \end{aligned}$$

$$\boxed{x_c = \frac{3}{8}\,a \qquad y_c = a} \quad (4)$$ Ans.

II. Second moments of area

Using thin-walled assumption

Section	$I_{x_{c\,i}}$	$I_{y_{c\,i}}$	$I_{x_c\,y_{c\,i}}$
12	$(2a\,t^3)/12 \approx 0$	$\left(t\,(2a)^3\right)/12$	0
23	$\left(2t\,(2a)^3\right)/12$	$\left(2a\,(2t)^3\right)/12 \approx 0$	0
34	$\left(a\,(2t)^3\right)/12 \approx 0$	$\left(2t\,(a)^3\right)/12$	0

Second Moment of area I_{xx} using thin-walled assumption is

$$\begin{aligned}
I_{xx12} &= I_{xc12} + A_{12}\left[dy_{12}\right]^2 \\
&= 0 + (2a)(t)\left[-a\right]^2 \\
I_{xx23} &= I_{xc23} + A_{23}\left[0\right]^2 \\
&= \frac{16}{12}\,t\,a^3 \\
I_{xx34} &= I_{xc34} + A_{34}\left[dy_{34}\right]^2 \\
&= 0 + (a)(2t)\left[a\right]^2 \\
I_{xx} &= I_{xx12} + I_{xx23} + I_{xx34} \\
&= 2\,t\,a^3 + \frac{4}{3}\,t\,a^3 + 2\,t\,a^3 = \frac{16}{3}\,t\,a^3
\end{aligned}$$

Second Moment of area I_{yy} using thin-walled assumption is

$$\begin{aligned}
I_{yy12} &= I_{y_{c12}} + A_{12}\left[dx_{12}\right]^2 \\
&= \frac{8}{12}\,t\,a^3 + (2a)(t)\left[\frac{5}{8}\,a\right]^2 \\
I_{yy23} &= I_{y_{c23}} + A_{23}\left[dx_{23}\right]^2 \\
&= 0 + (2a)(2t)\left[-\frac{3}{8}\,a\right]^2 \\
I_{yy34} &= I_{y_{c34}} + A_{34}\left[dx_{34}\right]^2 \\
&= \frac{2}{12}\,t\,a^3 + (a)(2t)\left[\frac{1}{8}\,a\right]^2 \\
I_{yy} &= I_{yy12} + I_{yy23} + I_{yy34} \\
&= \left(\frac{8}{12}\,t\,a^3 + \frac{50}{64}\,t\,a^3\right) + \left(0 + \frac{36}{64}\,t\,a^3\right) + \left(\frac{1}{6}\,t\,a^3 + \frac{2}{64}\,t\,a^3\right) = \frac{53}{24}\,t\,a^3
\end{aligned}$$

Second Moment of area I_{xy} using thin-walled assumption is

$$\begin{aligned} I_{xy12} &= I_{x_c\,y_{c12}} + A_{12}\,[dx_{12}\,dy_{12}] \\ &= 0 + (2a)(t)\left[\left(\frac{5}{8}\,a\right)(-a)\right] \end{aligned}$$

$$\begin{aligned} I_{xy23} &= I_{x_c\,y_{c23}} + A_{23}\,[dx_{23}\,dy_{23}] \\ &= 0 + (2a)(2t)\left[\left(-\frac{3}{8}\,a\right)(0)\right] \end{aligned}$$

$$\begin{aligned} I_{xy34} &= I_{x_c\,y_{c34}} + A_{34}\,[dx_{34}\,dy_{34}] \\ &= 0 + (a)(2t)\left[\left(\frac{1}{8}\,a\right)(a)\right] \end{aligned}$$

$$\begin{aligned} I_{xy} &= I_{xy12} + I_{xy23} + I_{xy34} \\ &= \frac{2}{8}\,t\,a^3 + 0 - \frac{10}{8}\,t\,a^3 = -t\,a^3 \end{aligned}$$

Second moments of area about the centroid

$$I_{xx} = \frac{16}{3}\,t\,a^3 \qquad I_{yy} = \frac{53}{24}\,t\,a^3 \qquad I_{xy} = -t\,a^3 \tag{5}$$

The factor in the denominator is

$$I_{xx}\,I_{yy} - I_{xy}^2 = \frac{97}{9}\,a^6\,t^2 \tag{6}$$

Ans.

III. Shear flow distribution and vertical location, η_s, of the shear center by considering $S_y = 0$ and $S_x > 0$

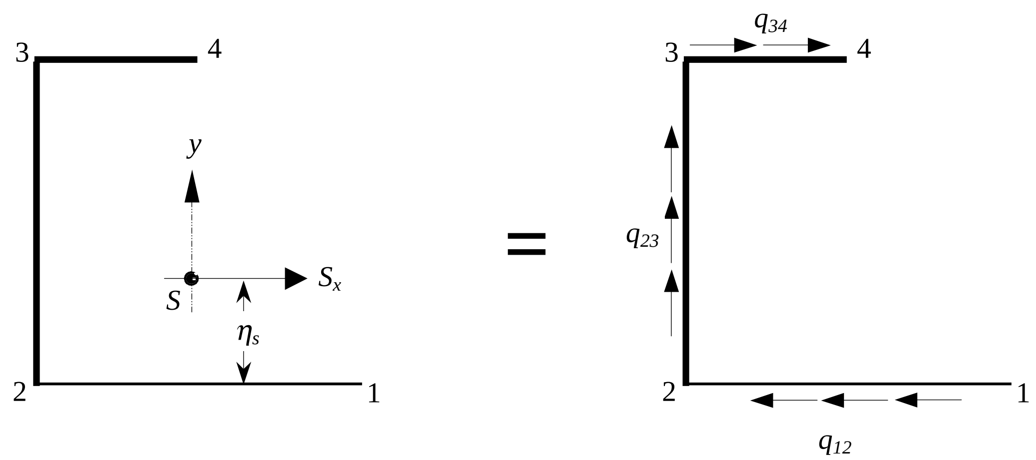

Fig. 5 Shear flow convention for statically equivalence when taking the torque at point 3

> When $S_y = 0$ and $S_x > 0$ is applied through the shear center then no torsion exists and the shear flow distribution for this problem is given by
>
> $$q_i(s) = q_{o_i} - \frac{S_x}{I_{xx}\, I_{yy} - I_{xy}^2} \left(I_{xx} \int_0^s t_i\, x_i(s_i)\, ds_i - I_{xy} \int_0^s t_i\, y_i(s_i)\, ds_i \right) \tag{7}$$
>
> where q_{o_i} is found by evaluating $q_i(s = 0)$.

Substitute the moments of area in the above equation to get

$$\begin{aligned} q_i(s) &= q_{o_i} - \frac{9\, S_x}{97\, a^6\, t^2} \left(\frac{16}{3}\, t\, a^3 \int_0^s t_i\, x_i(s_i)\, ds_i - (-t\, a^3) \int_0^s t_i\, y_i(s_i)\, ds_i \right) \qquad (8) \\ &= q_{o_i} - \frac{9\, S_x}{97\, a^3\, t} \left(\frac{16}{3} \int_0^s t_i\, x_i(s_i)\, ds_i + \int_0^s t_i\, y_i(s_i)\, ds_i \right) \end{aligned}$$

In order to calculate the shear flow in each flange, we need to first find the $y_i(s_i)$ for each part.

Following the assumed flow convention in Fig. 5 and from the geometry of the cross-section we get: (about the centroid)

For flange 12,

$$s_{12} = 0 \qquad \Rightarrow \qquad x_{12} = 2\,a - \frac{3}{8}\,a; \qquad y_{12} = -a \tag{9}$$

$$s_{12} = 2a \qquad \Rightarrow \qquad x_{12} = -\frac{3}{8}\,a; \qquad y_{12} = -a \tag{10}$$

For flange 23,

$$s_{23} = 0 \qquad \Rightarrow \qquad x_{23} = -\frac{3}{8}\,a; \qquad y_{23} = -a \tag{11}$$

$$s_{23} = 2a \qquad \Rightarrow \qquad x_{23} = -\frac{3}{8}\,a; \qquad y_{23} = a \tag{12}$$

For flange 34,

$$s_{34} = 0 \qquad \Rightarrow \qquad x_{34} = -\frac{3}{8}\,a; \qquad y_{34} = a \tag{13}$$

$$s_{34} = a \qquad \Rightarrow \qquad x_{34} = a - \frac{3}{8}\,a; \qquad y_{34} = a \tag{14}$$

Since these are straight lines, the equation of the line to can be used to calculate $x_i(s_i)$ and $y_i(s_i)$,

$$y_i(s_i) = \frac{y_i\Big|_{(s_i=a_i)} - y_i\Big|_{(s_i=0)}}{a_i - 0}\, s_i + b_i \qquad \text{where} \qquad b_i = y_i\Big|_{(s_i=0)} \tag{15}$$

$$x_i(s_i) = \frac{x_i\Big|_{(s_i=a_i)} - x_i\Big|_{(s_i=0)}}{a_i - 0}\, s_i + c_i \qquad \text{where} \qquad c_i = x_i\Big|_{(s_i=0)} \tag{16}$$

where a_i is the length of each branch.

For flange 12, $(t_{12} = t)$

$$y_{12}(s_{12}) = -a$$

$$x_{12}(s_{12}) = \frac{13}{8}\,a - s_{12}$$

$$\begin{aligned} q_{12}(s) &= q_{o_{12}} - \frac{9\,S_x}{97\,a^3\,t}\left(\frac{16}{3}\int_0^s t_{12}\,x_{12}(s_{12})\,ds_{12} + \int_0^s t_{12}\,y_{12}(s_{12})\,ds_{12}\right) \\ &= q_{o_{12}} - \frac{9\,S_x}{97\,a^3\,t}\left(\frac{16}{3}\int_0^s t\left(\frac{13}{8}\,a - s_{12}\right)ds_{12} + \int_0^s t\,(-a)\,ds_{12}\right) \\ &= q_{o_{12}} - \frac{69\,S_x\,s}{97\,a^2} + \frac{24\,S_x\,s^2}{97\,a^3} \end{aligned} \tag{17}$$

$$q_{o_{12}} = q_{12}(0) = 0 \qquad \text{free edge}$$

$$q_{12}(s) = -\frac{69\,S_x\,s}{97\,a^2} + \frac{24\,S_x\,s^2}{97\,a^3} \tag{18}$$

For flange 23, $(t_{23} = 2t)$

$$
\begin{aligned}
y_{23}(s_{23}) &= -a + s_{23} \\
x_{23}(s_{23}) &= -\frac{3}{8}\,a \\
q_{23}(s) &= q_{o_{23}} - \frac{9\,S_x}{97\,a^3\,t}\left(\frac{16}{3}\int_0^s t_{23}\,x_{23}(s_{23})\,ds_{23} + \int_0^s t_{23}\,y_{23}(s_{23})\,ds_{23}\right) \qquad (19) \\
&= q_{o_{23}} - \frac{9\,S_x}{97\,a^3\,t}\left(\frac{16}{3}\int_0^s 2t\left(-\frac{3}{8}\,a\right)ds_{23} + \int_0^s 2t\,(-a + s_{23})\,ds_{23}\right) \\
&= q_{o_{23}} + \frac{54\,S_x\,s}{97\,a^2} - \frac{9\,S_x\,s^2}{97\,a^3} \\
q_{o_{23}} &= q_{23}(0) = q_{12}(2a) = -\frac{42\,S_x}{97\,a} \\
q_{23}(s) &= -\frac{42\,S_x}{97\,a} + \frac{54\,S_x\,s}{97\,a^2} - \frac{9\,S_x\,s^2}{97\,a^3} \qquad (20)
\end{aligned}
$$

For flange 34, $(t_{34} = 2t)$

$$
\begin{aligned}
y_{34}(s_{34}) &= a \\
x_{34}(s_{34}) &= -\frac{3}{8}\,a + s_{34} \\
q_{34}(s) &= q_{o_{34}} - \frac{9\,S_x}{97\,a^3\,t}\left(\frac{16}{3}\int_0^s t_{34}\,x_{34}(s_{34})\,ds_{34} + \int_0^s t_{34}\,y_{34}(s_{34})\,ds_{34}\right) \qquad (21) \\
&= q_{o_{34}} - \frac{9\,S_x}{97\,a^3\,t}\left(\frac{16}{3}\int_0^s 2t\left(-\frac{3}{8}\,a + s_{34}\right)ds_{34} + \int_0^s 2t\,(a)\,ds_{34}\right) \\
&= q_{o_{34}} + \frac{18\,S_x\,s}{97\,a^2} - \frac{48\,S_x\,s^2}{97\,a^3} \\
q_{o_{34}} &= q_{34}(0) = q_{23}(2a) = \frac{30\,S_x}{97\,a} \\
q_{34}(s) &= \frac{30\,S_x}{97\,a} + \frac{18\,S_x\,s}{97\,a^2} - \frac{48\,S_x\,s^2}{97\,a^3} \qquad (22)
\end{aligned}
$$

Check: Since point 4 is a free edge

$$
\begin{aligned}
q_{34}(a) &= \text{must be zero} \\
&= \frac{30\,S_x}{97\,a} + \frac{18\,S_x\,a}{97\,a^2} - \frac{48\,S_x\,a^2}{97\,a^3} \\
&= 0 \text{ ...GOOD!}
\end{aligned}
$$

As mentioned before, for torque equivalence about a point 3, $O_s = 3$, with $S_y = 0$ and $S_x > 0$,

$$S_x\,(2a - \eta_s) = \sum_i \int_0^{a_i} \overrightarrow{r}_i(s) \times \overrightarrow{q}_i(s)\,ds \tag{23}$$

$$S_x\,(2a - \eta_s) = -\int_0^{a_{12}} r_{12}(s)\,q_{12}(s)\,ds \tag{24}$$

$$\tag{25}$$

For this problem, $r_{12}(s) = 2a$ and $a_{12} = 2a$:

$$S_x\,(2a - \eta_s) = -\int_0^{2a} 2a\left(-\frac{69\,S_x\,s}{97\,a^2} + \frac{24\,S_x\,s^2}{97\,a^3}\right)ds \tag{26}$$

$$= \frac{148\,S_x\,a}{97} \tag{27}$$

$$\boxed{\eta_s = \frac{46}{97}\,a}$$ Ans.

IV. Shear flow distribution and horizontal location, ξ_s, of the shear center by considering $S_x = 0$ and $S_y > 0$

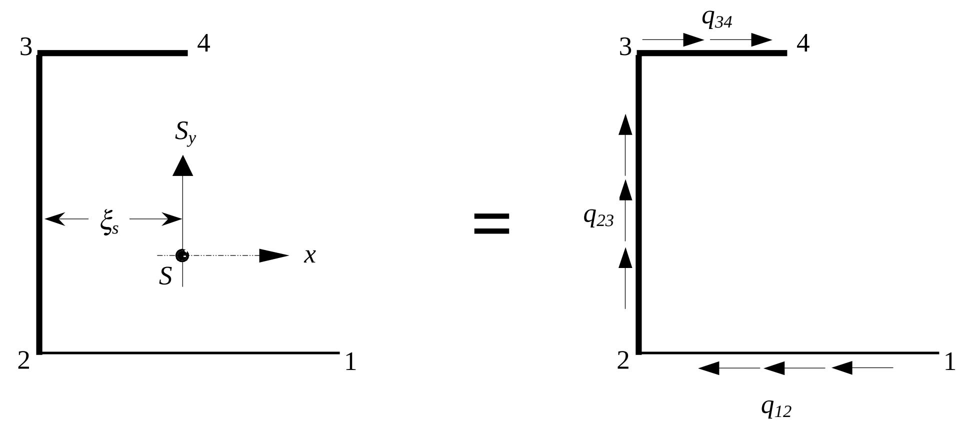

Fig. 6 Shear flow convention for statically equivalence when taking the torque at point 3

When $S_x = 0$ and $S_y > 0$ is applied through the shear center then no torsion exists and the shear flow distribution is given by

$$q_i(s) = q_{o_i} - \frac{S_y}{I_{xx}\, I_{yy} - I_{xy}^2} \left(-I_{xy} \int_0^s t_i(s_i)\, x_i(s_i)\, ds_i + I_{yy} \int_0^s t_i(s_i)\, y_i(s_i)\, ds_i \right) \tag{28}$$

where q_{o_i} is found by evaluating $q_i(s = 0)$.

Substitute the moments of area in the above equation to get

$$\begin{aligned} q_i(s) &= q_{o_i} - \frac{9\, S_y}{97\, a^6\, t^2} \left(-(-t\, a^3) \int_0^s t_i\, x_i(s_i)\, ds_i + \frac{53}{24}\, t\, a^3 \int_0^s t_i\, y_i(s_i)\, ds_i \right) \\ &= q_{o_i} - \frac{9\, S_y}{97\, a^3\, t} \left(\int_0^s t_i\, x_i(s_i)\, ds_i + \frac{53}{24} \int_0^s t_i\, y_i(s_i)\, ds_i \right) \end{aligned} \tag{29}$$

In order to calculate the shear flow in each flange, we need to first find the $y_i(s_i)$ for each part. However, these were found previously.

For flange 12, $(t_{12} = t)$

$$
\begin{aligned}
y_{12}(s_{12}) &= -a \\
x_{12}(s_{12}) &= \frac{13}{8}\,a - s_{12} \\
q_{12}(s) &= q_{o_{12}} - \frac{9\,S_y}{97\,a^3\,t}\left(\int_0^s t_{12}\,y_{12}(s_{12})\,ds_{12} + \frac{53}{24}\int_0^s t_{12}\,x_{12}(s_{12})\,ds_{12}\right) \qquad (30) \\
&= q_{o_{12}} + \frac{21\,S_y\,s}{388\,a^2} + \frac{9\,S_y\,s^2}{194\,a^3} \\
q_{o_{12}} &= q_{12}(0) = 0 \qquad \text{free edge} \\
q_{12}(s) &= \frac{21\,S_y\,s}{388\,a^2} + \frac{9\,S_y\,s^2}{194\,a^3} \qquad (31)
\end{aligned}
$$

For flange 23, $(t_{23} = 2t)$

$$
\begin{aligned}
y_{23}(s_{23}) &= -a + s_{23} \\
x_{23}(s_{23}) &= -\frac{3}{8}\,a \\
q_{23}(s) &= q_{o_{23}} - \frac{9\,S_y}{97\,a^3\,t}\left(\int_0^s t_{23}\,y_{23}(s_{23})\,ds_{23} + \frac{53}{24}\int_0^s t_{23}\,x_{23}(s_{23})\,ds_{23}\right) \qquad (32) \\
&= q_{o_{23}} + \frac{93\,S_y\,s}{194\,a^2} - \frac{159\,S_y\,s^2}{776\,a^3} \\
q_{o_{23}} &= q_{23}(0) = q_{12}(2a) = -\frac{57\,S_y}{194\,a} \\
q_{23}(s) &= -\frac{57\,S_y}{194\,a} + \frac{93\,S_y\,s}{194\,a^2} - \frac{159\,S_y\,s^2}{776\,a^3} \qquad (33)
\end{aligned}
$$

For flange 34, $(t_{34} = 2t)$

$$
\begin{aligned}
y_{34}(s_{34}) &= a \\
x_{34}(s_{34}) &= -\frac{3}{8}\,a + s_{34} \\
q_{34}(s) &= q_{o_{34}} - \frac{9\,S_y}{97\,a^3\,t}\left(\int_0^s t_{34}\,y_{34}(s_{34})\,ds_{34} + \frac{53}{24}\int_0^s t_{34}\,x_{34}(s_{34})\,ds_{34}\right) \qquad (34) \\
&= q_{o_{34}} - \frac{33\,S_y\,s}{97\,a^2} - \frac{9\,S_y\,s^2}{97\,a^3} \\
q_{o_{34}} &= q_{34}(0) = q_{23}(2a) = \frac{42\,S_y}{97\,a} \\
q_{34}(s) &= \frac{42\,S_y}{97\,a} - \frac{33\,S_y\,s}{97\,a^2} - \frac{9\,S_y\,s^2}{97\,a^3} \qquad (35)
\end{aligned}
$$

Check: Since point 4 is a free edge

$$\begin{aligned} q_{34}(a) &= \text{must be zero} \\ &= \frac{42\, S_y}{97\, a} - \frac{33\, S_y\, a}{97\, a^2} - \frac{9\, S_y\, a^2}{97\, a^3} \\ &= 0 \text{ ...GOOD!} \end{aligned}$$

As mentioned before, for torque equivalence about a point 3, $O_s = 3$, with $S_x = 0$ and $S_y > 0$,

$$S_y\, \xi_s = \sum_i \int_0^{a_i} \overrightarrow{r}_i(s) \times \overrightarrow{q}_i(s)\, ds \quad (36)$$

$$S_y\, \xi_s = -\int_0^{a_{12}} r_{12}(s)\, q_{12}(s)\, ds \quad (37)$$

$$(38)$$

For this problem, $r_{12}(s) = 2a$ and $a_{12} = 2a$:

$$S_y\, \xi_s = -\int_0^{2a} 2a \left(\frac{21\, S_y\, s}{388\, a^2} + \frac{9\, S_y\, s^2}{194\, a^3} \right) ds \quad (39)$$

$$= -\frac{47\, S_7\, a}{97} \quad (40)$$

$$\boxed{\xi_s = -\frac{45}{97}\, a}$$

Ans.