AOE 3024: Thin Walled Structures
Solutions to Homework # 12

Determine displacement at each node and the axial force in each spring.

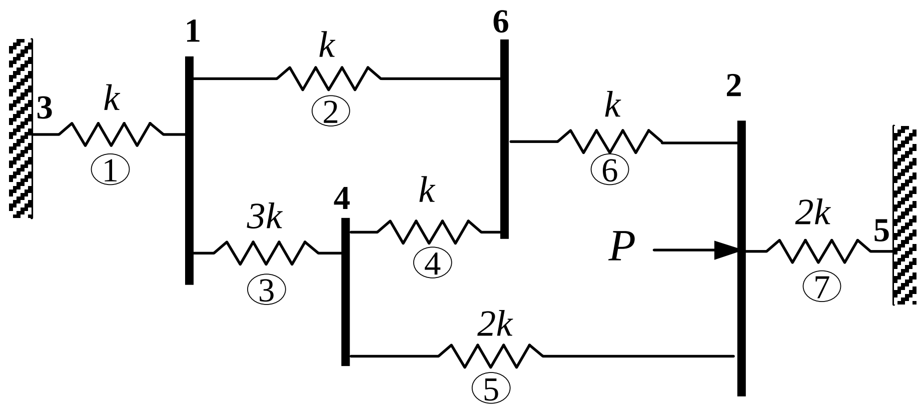


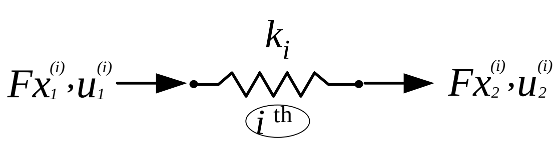
Fig. 1 Problem 12

The force displacement relationship for one element is

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(i)} = \begin{bmatrix} k_i & -k_i \\ -k_i & k_i \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^{(i)} \tag{1}$$

Note that the force can also be calculated by

$$F_{x_1}^{(i)} = k_i \left(u_1^{(i)} - u_2^{(i)} \right) \qquad F_{x_2}^{(i)} = k_i \left(u_2^{(i)} - u_1^{(i)} \right) \tag{2}$$



Element 1

$$u_1^{(1)} = u_3 = 0 \qquad u_2^{(1)} = u_1 \tag{3}$$

Thus the connectivity vector is: $C_1 = \{3, 1\}$,

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(1)} = \begin{array}{c} \\ 3 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 3 & 1 \end{array} \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array} \begin{Bmatrix} u_1^{(1)} = u_3 = 0 \\ u_2^{(1)} = u_1 \end{Bmatrix} \tag{4}$$

where $k_1 = k$.

Element 2

$$u_1^{(2)} = u_1 \qquad u_2^{(2)} = u_6 \tag{5}$$

Thus the connectivity vector is: $C_2 = \{1, 6\}$,

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(2)} = \begin{matrix} \\ 1 \\ 6 \end{matrix} \begin{matrix} \begin{matrix} 1 & \quad 6 \end{matrix} \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{matrix} \begin{Bmatrix} u_1^{(2)} = u_1 \\ u_2^{(2)} = u_6 \end{Bmatrix} \tag{6}$$

where $k_2 = k$.

Element 3

$$u_1^{(3)} = u_1 \qquad u_2^{(3)} = u_4 \tag{7}$$

Thus the connectivity vector is: $C_3 = \{1, 4\}$,

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(3)} = \begin{matrix} \\ 1 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 1 & \quad 4 \end{matrix} \\ \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \end{matrix} \begin{Bmatrix} u_1^{(3)} = u_1 \\ u_2^{(3)} = u_4 \end{Bmatrix} \tag{8}$$

where $k_3 = 3\,k$.

Element 4

$$u_1^{(4)} = u_4 \qquad u_2^{(4)} = u_6 \tag{9}$$

Thus the connectivity vector is: $C_4 = \{4, 6\}$,

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(4)} = \begin{matrix} \\ 4 \\ 6 \end{matrix} \begin{matrix} \begin{matrix} 4 & \quad 6 \end{matrix} \\ \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} \end{matrix} \begin{Bmatrix} u_1^{(4)} = u_4 \\ u_2^{(4)} = u_6 \end{Bmatrix} \tag{10}$$

where $k_4 = k$.

Element 5

$$u_1^{(5)} = u_4 \qquad u_2^{(5)} = u_2 \tag{11}$$

Thus the connectivity vector is: $C_5 = \{4, 2\}$,

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(4)} = \begin{matrix} \\ 4 \\ 2 \end{matrix} \begin{matrix} \begin{matrix} 4 & \quad 2 \end{matrix} \\ \begin{bmatrix} k_5 & -k_5 \\ -k_5 & k_5 \end{bmatrix} \end{matrix} \begin{Bmatrix} u_1^{(5)} = u_4 \\ u_2^{(5)} = u_2 \end{Bmatrix} \tag{12}$$

where $k_5 = 2\,k$.

Element 6

$$u_1^{(6)} = u_6 \qquad u_2^{(6)} = u_2 \tag{13}$$

Thus the connectivity vector is: $C_6 = \{6, 2\}$,

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(6)} = \begin{matrix} \\ 6 \\ 2 \end{matrix} \quad \begin{matrix} \begin{matrix} 6 & \quad 2 \end{matrix} \\ \begin{bmatrix} k_6 & -k_6 \\ -k_6 & k_6 \end{bmatrix} \end{matrix} \begin{Bmatrix} u_1^{(6)} = u_6 \\ u_2^{(6)} = u_2 \end{Bmatrix} \tag{14}$$

where $k_6 = k$.

Element 7

$$u_1^{(7)} = u_2 \qquad u_2^{(7)} = u_5 = 0 \tag{15}$$

Thus the connectivity vector is: $C_7 = \{2, 5\}$,

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(7)} = \begin{matrix} \\ 2 \\ 5 \end{matrix} \quad \begin{matrix} \begin{matrix} 2 & \quad 5 \end{matrix} \\ \begin{bmatrix} k_7 & -k_7 \\ -k_7 & k_7 \end{bmatrix} \end{matrix} \begin{Bmatrix} u_1^{(7)} = u_2 \\ u_2^{(7)} = u_5 = 0 \end{Bmatrix} \tag{16}$$

where $k_7 = 2\,k$.

Assemblage is done by using the connectivity vectors:

$$\begin{Bmatrix} F_{x_1} = 0 \\ F_{x_2} = P \\ F_{x_3} = ?? \\ F_{x_4} = 0 \\ F_{x_5} = ?? \\ F_{x_6} = 0 \end{Bmatrix} =$$

$$\begin{matrix} . \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} k_1 + k_2 + k_3 & 0 & -k_1 & -k_3 & 0 & -k_2 \\ 0 & k_5 + k_6 + k_7 & 0 & -k_5 & -k_7 & -k_6 \\ -k_1 & 0 & k_1 & 0 & 0 & 0 \\ -k_3 & -k_5 & 0 & k_3 + k_4 + k_5 & 0 & -k_4 \\ 0 & -k_7 & 0 & 0 & k_7 & 0 \\ -k_2 & -k_6 & 0 & -k_4 & 0 & k_2 + k_4 + k_6 \end{bmatrix} \end{matrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 = 0 \\ u_4 \\ u_5 = 0 \\ u_6 \end{Bmatrix} \tag{17}$$

Substituting values all stiffness values we get,

$$\begin{Bmatrix} 0 \\ P \\ Q_{x3} \\ 0 \\ Q_{x5} \\ 0 \end{Bmatrix} = k \underbrace{\begin{bmatrix} 5 & 0 & -1 & -3 & 0 & -1 \\ 0 & 5 & 0 & -2 & -2 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -3 & -2 & 0 & 6 & 0 & -1 \\ 0 & -2 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & -1 & 0 & 3 \end{bmatrix}}_{\text{Stiffness Matrix: } [K]} \begin{Bmatrix} u_1 \\ u_2 \\ 0 \\ u_4 \\ 0 \\ u_6 \end{Bmatrix}$$

We Recall that the stiffness matrix for linear and conservative problems is symmetric and singular. Also, at any given point either prescribe displacement or load but not both.

The following displacements are prescribed: $u_3 = 0$, $u_5 = 0$. Thus we can eliminate the rows and columns associated with these displacements

$$\underbrace{\begin{Bmatrix} 0 \\ P \\ 0 \\ 0 \end{Bmatrix}}_{\{F_g\}} = k \underbrace{\begin{bmatrix} 5 & 0 & -3 & -1 \\ 0 & 5 & -2 & -1 \\ -3 & -2 & 6 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}}_{[K_g]} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_4 \\ u_6 \end{Bmatrix}}_{\{u_g\}}$$

The displacement are found as follows,

$$\mathbf{F_g} = \mathbf{K_g u_g} \qquad \Rightarrow \qquad \mathbf{u_g} = \mathbf{K_g^{-1} F_g} \tag{18}$$

where the inverse of the reduced stiffness matrix is (see Mathematica file for details)

$$\mathbf{K_g^{-1}} = \frac{1}{121\,k} \begin{bmatrix} 63 & 29 & 49 & 47 \\ 29 & 46 & 36 & 37 \\ 49 & 36 & 65 & 50 \\ 47 & 37 & 50 & 85 \end{bmatrix} \tag{19}$$

Thus

$$\mathbf{u_g} = \frac{1}{121\,k} \begin{bmatrix} 63 & 29 & 49 & 47 \\ 29 & 46 & 36 & 37 \\ 49 & 36 & 65 & 50 \\ 47 & 37 & 50 & 85 \end{bmatrix} \begin{Bmatrix} 0 \\ P \\ 0 \\ 0 \end{Bmatrix} \tag{20}$$

The nodal displacements are

$$u_1 = \frac{29}{121}\frac{P}{k} \qquad u_2 = \frac{46}{121}\frac{P}{k} \qquad u_4 = \frac{36}{121}\frac{P}{k} \qquad u_6 = \frac{37}{121}\frac{P}{k} \qquad u_3 = u_5 = 0$$

Ans.

The axial forces are found using the element matrices. See Mathematica file for details.

Element 1

$$u_1^{(1)} = u_3 = 0 \qquad u_2^{(1)} = u_1 = \frac{29}{121}\frac{P}{k} \tag{21}$$

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(1)} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_3 \\ u_1 \end{Bmatrix} = \frac{P}{121} \begin{Bmatrix} -29 \\ 29 \end{Bmatrix} \tag{22}$$

Axial forces for spring element 1 are

$$F_{x_1}^{(1)} = -\frac{29}{121}\,P \qquad F_{x_2}^{(1)} = \frac{29}{121}\,P$$

Ans.

Element 2

$$u_1^{(2)} = u_1 = \frac{29}{121}\frac{P}{k} \qquad u_2^{(2)} = u_6 = \frac{37}{121}\frac{P}{k} \tag{23}$$

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(2)} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_6 \end{Bmatrix} = \frac{P}{121} \begin{Bmatrix} -8 \\ 8 \end{Bmatrix} \tag{24}$$

Axial forces for spring element 2 are

$$F_{x_1}^{(2)} = -\frac{8}{121}\,P \qquad F_{x_2}^{(2)} = \frac{8}{121}\,P$$

Ans.

Element 3

$$u_1^{(3)} = u_1 = \frac{29}{121}\frac{P}{k} \qquad u_2^{(3)} = u_4 = \frac{36}{121}\frac{P}{k} \tag{25}$$

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(3)} = \begin{bmatrix} 3k & -3k \\ -3k & 3k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_4 \end{Bmatrix} = \frac{P}{121} \begin{Bmatrix} -21 \\ 21 \end{Bmatrix} \tag{26}$$

Axial forces for spring element 3 are

$$F_{x_1}^{(3)} = -\frac{21}{121}\,P \qquad F_{x_2}^{(3)} = \frac{21}{121}\,P$$

Ans.

Element 4

$$u_1^{(4)} = u_4 = \frac{36}{121}\frac{P}{k} \qquad u_2^{(4)} = u_6 = \frac{37}{121}\frac{P}{k} \tag{27}$$

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(4)} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} = \frac{P}{121} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \tag{28}$$

Axial forces for spring element 4 are

$$F_{x_1}^{(4)} = -\frac{1}{121}\,P \qquad F_{x_2}^{(4)} = \frac{1}{121}\,P$$

Ans.

Element 5

$$u_1^{(5)} = u_4 = \frac{36}{121}\frac{P}{k} \qquad u_2^{(5)} = u_2 = \frac{46}{121}\frac{P}{k} \tag{29}$$

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(4)} = \begin{bmatrix} 2k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} u_4 \\ u_2 \end{Bmatrix} = \frac{P}{121} \begin{Bmatrix} -20 \\ 20 \end{Bmatrix} \tag{30}$$

Axial forces for spring element 5 are

$$F_{x_1}^{(5)} = -\frac{20}{121}\,P \qquad F_{x_2}^{(5)} = \frac{20}{121}\,P$$

Ans.

Element 6

$$u_1^{(6)} = u_6 = \frac{37}{121}\frac{P}{k} \qquad u_2^{(6)} = u_2 = \frac{46}{121}\frac{P}{k} \tag{31}$$

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(6)} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_6 \\ u_2 \end{Bmatrix} = \frac{P}{121} \begin{Bmatrix} -9 \\ 9 \end{Bmatrix} \tag{32}$$

Axial forces for spring element 6 are

$$F_{x_1}^{(6)} = -\frac{9}{121}\,P \qquad F_{x_2}^{(6)} = \frac{9}{121}\,P$$

Ans.

Element 7

$$u_1^{(7)} = u_2 = \frac{46}{121}\frac{P}{k} \qquad u_2^{(7)} = u_5 = 0 \tag{33}$$

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \end{Bmatrix}^{(7)} = \begin{bmatrix} 2k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} u_2 \\ u_5 \end{Bmatrix} = \frac{P}{121} \begin{Bmatrix} 92 \\ -92 \end{Bmatrix} \tag{34}$$

Axial forces for spring element 7 are

$$F_{x_1}^{(7)} = \frac{92}{121}\,P \qquad F_{x_2}^{(7)} = -\frac{92}{121}\,P$$

Ans.