AOE 3024: Thin Walled Structures
Solution to Homework # 1

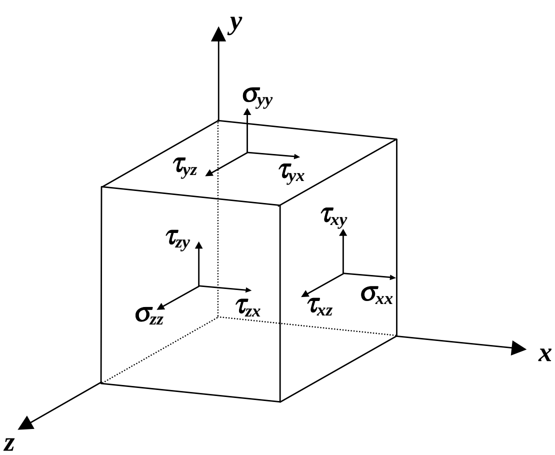
The state of stress at a point in a component is given as

$$\begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \text{ MPa} \tag{1}$$

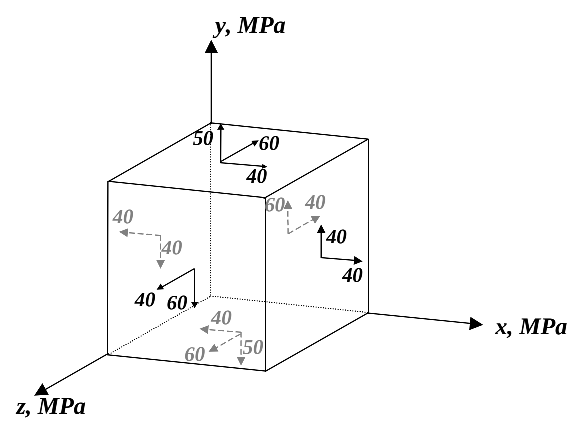
a. Show this state of stress on a differential element

The state of stress is expressed as follows

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \text{ MPa} \tag{2}$$



a) This figure shows the sign convention followed by Eq. (2) on the positive faces



b) The gray dashed arrows and numbers are on hidden faces

Fig. 1 This is an infinitesimal element representing the state of stress for the given problem (NOTE: Units are part of the answer)

b. Determine the stress vectors and the total force vectors acting on the faces OAC, OCB, and OBA. Note that $OA = 2OB = 2OC = \Delta$.

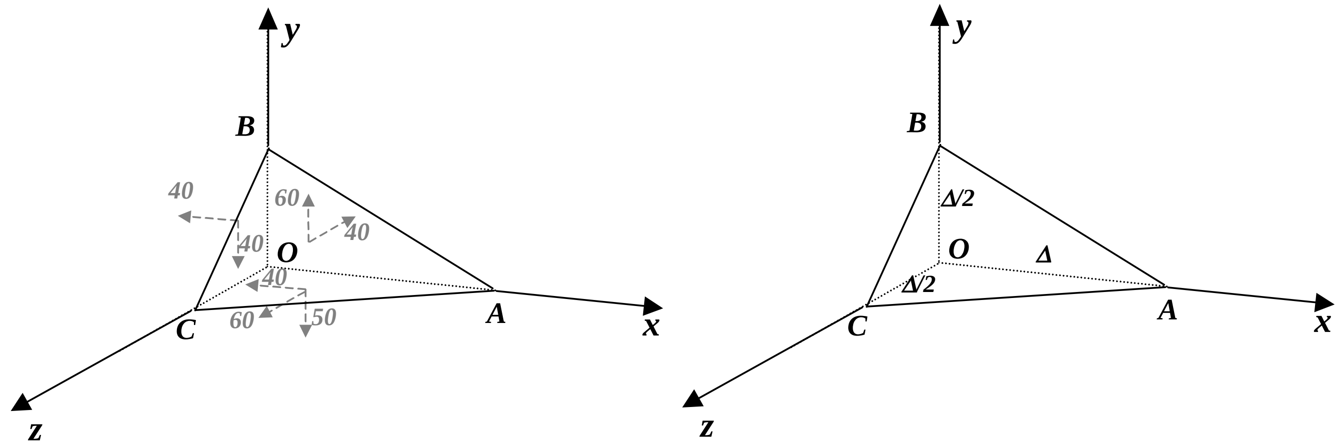


Fig. 2 Faces OAC, OCB, and OBA are shown with the acting stresses

Two methods exist to find the stress vectors: static equilibrium of stresses and Cauchy's relation (Cauchy's formula)

METHOD ONE: Using static equilibrium

The infinitesimal element in Fig. (1) is in equilibrium. Therefore, the sum of all stresses in any direction should be zero:

$$\overrightarrow{\mathbf{T}}^{(j)} + \overrightarrow{\mathbf{T}}^{(-j)} = 0 \;\Rightarrow\; \overrightarrow{\mathbf{T}}^{(-j)} = -\overrightarrow{\mathbf{T}}^{(j)} \text{ where } j \text{ represents the direction} \tag{3}$$

Stress vector on face OBA

$$\overrightarrow{\mathbf{T}}^{OBA} = -\tau_{zx}\,\hat{\mathbf{i}} - \tau_{zy}\,\hat{\mathbf{j}} - \sigma_{zz}\hat{\mathbf{k}} \tag{4a}$$

$$= -(0)\,\hat{\mathbf{i}} - (-60)\,\hat{\mathbf{j}} - (40)\hat{\mathbf{k}} \text{ MPa} \tag{4b}$$

$$= 60\,\hat{\mathbf{j}} - 40\,\hat{\mathbf{k}} \text{ MPa} \tag{4c}$$

Stress vector on face OCB

$$\overrightarrow{\mathbf{T}}^{OCB} = -\sigma_{xx}\,\hat{\mathbf{i}} - \tau_{xy}\,\hat{\mathbf{j}} - \tau_{xz}\hat{\mathbf{k}} \tag{5a}$$

$$= -(40)\,\hat{\mathbf{i}} - (40)\,\hat{\mathbf{j}} - (0)\hat{\mathbf{k}} \text{ MPa} \tag{5b}$$

$$= -40\,\hat{\mathbf{i}} - 40\,\hat{\mathbf{j}} \text{ MPa} \tag{5c}$$

Stress vector on face OAC

$$\overrightarrow{\mathbf{T}}^{OAC} = -\tau_{yx}\,\hat{\mathbf{i}} - \sigma_{yy}\,\hat{\mathbf{j}} - \tau_{yz}\hat{\mathbf{k}} \tag{6a}$$

$$= -(40)\,\hat{\mathbf{i}} - (50)\,\hat{\mathbf{j}} - (-60)\hat{\mathbf{k}} \text{ MPa} \tag{6b}$$

$$= -40\,\hat{\mathbf{i}} - 50\,\hat{\mathbf{j}} + 60\,\hat{\mathbf{k}} \text{ MPa} \tag{6c}$$

METHOD TWO: Using Cauchy's Relation

Find unit vectors on faces on which traction forces are desired

$$\hat{\mathbf{n}}_{OAC} = -\hat{\mathbf{j}} \tag{7}$$
$$\hat{\mathbf{n}}_{OCB} = -\hat{\mathbf{i}} \tag{8}$$
$$\hat{\mathbf{n}}_{OBA} = -\hat{\mathbf{k}} \tag{9}$$

Stress vectors are found using Cauchy's formula

$$\overrightarrow{\mathbf{T}}^{j} = [\boldsymbol{\sigma}] \cdot \hat{\mathbf{n}}_j \tag{10}$$

where $[\boldsymbol{\sigma}]$ is given by Eq. (2).

Stress vector on face OBA

$$\overrightarrow{\mathbf{T}}^{OBA} = [\boldsymbol{\sigma}] \cdot \hat{\mathbf{n}}_{OBA} \tag{11a}$$

$$= \begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60 \\ -40 \end{Bmatrix} \text{ MPa} \tag{11b}$$

$$= 60\,\hat{\mathbf{j}} - 40\,\hat{\mathbf{k}} \text{ MPa} \tag{11c}$$

Stress vector on face OCB

$$\overrightarrow{\mathbf{T}}^{OCB} = [\boldsymbol{\sigma}] \cdot \hat{\mathbf{n}}_{OCB} \tag{12a}$$

$$= \begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \cdot \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -40 \\ -40 \\ 0 \end{Bmatrix} \text{ MPa} \tag{12b}$$

$$= -40\,\hat{\mathbf{i}} - 40\,\hat{\mathbf{j}} \text{ MPa} \tag{12c}$$

Stress vector on face OAC

$$\overrightarrow{\mathbf{T}}^{OAC} = [\boldsymbol{\sigma}] \cdot \hat{\mathbf{n}}_{OAC} \tag{13a}$$

$$= \begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -40 \\ -50 \\ 60 \end{Bmatrix} \text{ MPa} \tag{13b}$$

$$= -40\,\hat{\mathbf{i}} - 50\,\hat{\mathbf{j}} + 60\,\hat{\mathbf{k}} \text{ MPa} \tag{13c}$$

Since force is equal to stress multiplied by area, we proceed to calculate the area of faces OAC, OCB, OBA

$$\begin{aligned} A &= \frac{1}{2} b\, h \\ A_{OAC} &= \frac{1}{2}(\Delta)\left(\frac{\Delta}{2}\right) = \frac{1}{4}\Delta^2 && (14) \\ A_{OCB} &= \frac{1}{2}\left(\frac{\Delta}{2}\right)\left(\frac{\Delta}{2}\right) = \frac{1}{8}\Delta^2 && (15) \\ A_{OBA} &= \frac{1}{2}(\Delta)\left(\frac{\Delta}{2}\right) = \frac{1}{4}\Delta^2 && (16) \end{aligned}$$

Total force acting on face OBA: *(Assumed that distance Δ is given in meters)*

$$\begin{aligned} \overrightarrow{\mathbf{F}}^{(OBA)} &= \overrightarrow{\mathbf{T}}^{(OBA)} A_{OBA} && (17a) \\ &= \frac{60}{4}\Delta^2\,\hat{\mathbf{j}} - \frac{40}{4}\Delta^2\,\hat{\mathbf{k}} \text{ MPa-m}^2 && (17b) \\ &= 15\Delta^2\,\hat{\mathbf{j}} - 10\Delta^2\,\hat{\mathbf{k}} \text{ MN} && (17c) \end{aligned}$$

Total force acting on face OCB: *(Assumed that distance Δ is given in meters)*

$$\begin{aligned} \overrightarrow{\mathbf{F}}^{(OCB)} &= \overrightarrow{\mathbf{T}}^{(OCB)} A_{OCB} && (18a) \\ &= -\frac{40}{8}\Delta^2\,\hat{\mathbf{i}} - \frac{40}{8}\Delta^2\,\hat{\mathbf{j}} \text{ MPa-m}^2 && (18b) \\ &= -5\Delta^2\,\hat{\mathbf{i}} - 5\Delta^2\,\hat{\mathbf{j}} \text{ MN} && (18c) \end{aligned}$$

Total force acting on face OAC: *(Assumed that distance Δ is given in meters)*

$$\begin{aligned} \overrightarrow{\mathbf{F}}^{(OAC)} &= \overrightarrow{\mathbf{T}}^{(OAC)} A_{OAC} && (19a) \\ &= -\frac{40}{4}\Delta^2\,\hat{\mathbf{i}} - \frac{50}{4}\Delta^2\,\hat{\mathbf{j}} + \frac{60}{4}\Delta^2\,\hat{\mathbf{k}} \text{ MPa-m}^2 && (19b) \\ &= -10\Delta^2\,\hat{\mathbf{i}} - 12.5\Delta^2\,\hat{\mathbf{j}} + 15\Delta^2\,\hat{\mathbf{k}} \text{ MN} && (19c) \end{aligned}$$

c. Determine the stress vector acting on the face ABC.

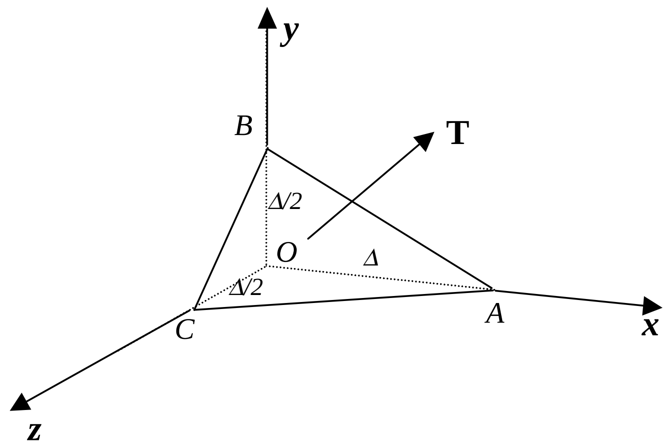


Fig. 3 Stress vector T acts on face ABC

METHOD ONE: STATIC EQUILIBRIUM

Since stresses are obtained dividing forces by the area, we proceed to find the area A_{ABC}

$$\overline{AB} = \sqrt{\left(\frac{\Delta}{2}\right)^2 + (\Delta)^2} = \frac{\sqrt{5}}{2}\Delta \tag{20}$$

$$\overline{BC} = \sqrt{\left(\frac{\Delta}{2}\right)^2 + \left(\frac{\Delta}{2}\right)^2} = \frac{\sqrt{2}}{2}\Delta \tag{21}$$

$$\overline{CA} = \sqrt{\left(\frac{\Delta}{2}\right)^2 + (\Delta)^2} = \frac{\sqrt{5}}{2}\Delta \tag{22}$$

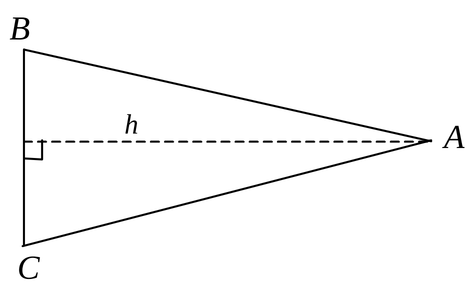


Fig. 4 The height of face ABC is assumed as h

$$h = \sqrt{-\left(\frac{\overline{BC}}{2}\right)^2 + \left(\overline{AB}\right)^2} \tag{23a}$$

$$= \sqrt{-\left(\frac{\sqrt{2}}{4}\Delta\right)^2 + \left(\frac{\sqrt{5}}{2}\Delta\right)^2} = \frac{3\sqrt{2}}{4}\Delta \tag{23b}$$

$$A_{ABC} = \frac{1}{2}\, b\, h = \frac{1}{2}\,\overline{BC}\, h \tag{24a}$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\Delta\right)\left(\frac{3\sqrt{2}}{4}\Delta\right) = \frac{3}{8}\Delta^2 \tag{24b}$$

Note that element $OABC$ is in static equilibrium, therefore

$$\sum \overrightarrow{\mathbf{F}} = 0 \Rightarrow \overrightarrow{\mathbf{F}}_{OAC} + \overrightarrow{\mathbf{F}}_{OCB} + \overrightarrow{\mathbf{F}}_{OBA} + \overrightarrow{\mathbf{F}}_{ABC} = 0 \tag{25}$$

Total force acting on face ABC is

$$\begin{aligned}
\overrightarrow{\mathbf{F}}_{ABC} &= -\overrightarrow{\mathbf{F}}_{OAC} - \overrightarrow{\mathbf{F}}_{OCB} - \overrightarrow{\mathbf{F}}_{OBA} && \text{(26a)}\\
&= -\left(15\Delta^2\,\hat{\mathbf{j}} - 10\Delta^2\,\hat{\mathbf{k}}\right) - \left(-5\Delta^2\,\hat{\mathbf{i}} - 5\Delta^2\,\hat{\mathbf{j}}\right) - \left(-10\Delta^2\,\hat{\mathbf{i}} - 12.5\Delta^2\,\hat{\mathbf{j}} + 15\Delta^2\,\hat{\mathbf{k}}\right)\ \text{MN} && \text{(26b)}\\
&= 15\Delta^2\,\hat{\mathbf{i}} + 2.5\Delta^2\,\hat{\mathbf{j}} - 5\Delta^2\,\hat{\mathbf{k}}\ \text{MN} && \text{(26c)}
\end{aligned}$$

Stress vector is obtain by dividing the total force acting on face ABC by area of face ABC

$$\begin{aligned}
\overrightarrow{\mathbf{T}} &= \frac{\overrightarrow{\mathbf{F}}_{ABC}}{A_{ABC}} && \text{(27a)}\\
&= \frac{15\Delta^2\,\hat{\mathbf{i}} + 2.5\Delta^2\,\hat{\mathbf{j}} - 5\Delta^2\,\hat{\mathbf{k}}\ \text{MN}}{\frac{3}{8}\Delta^2} && \text{(27b)}\\
&= 40\,\hat{\mathbf{i}} + \frac{20}{3}\,\hat{\mathbf{j}} - \frac{40}{3}\,\hat{\mathbf{k}}\ \text{MPa} && \text{(27c)}
\end{aligned}$$

METHOD TWO: CAUCHY'S FORMULA & PLANE ANALYTICAL GEOMETRY

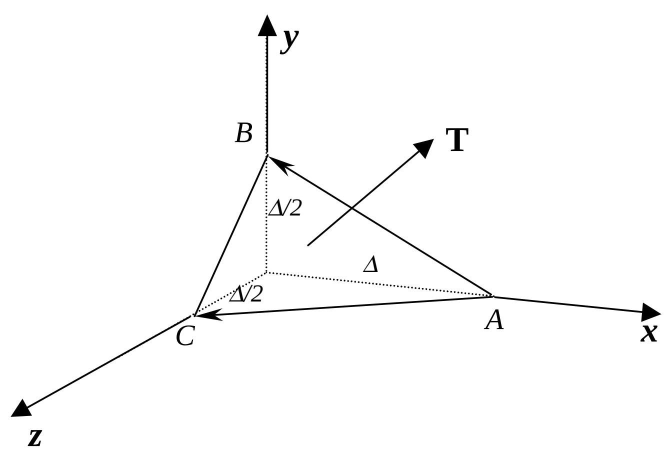


Fig. 5

Area of triangle ABC can be found by using the following equation

$$\begin{aligned}
\overrightarrow{\mathbf{A}}_{ABC} &= \frac{1}{2}\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}} && \text{(28a)}\\
&= \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\Delta & \Delta/2 & 0 \\ -\Delta & 0 & \Delta/2 \end{vmatrix} && \text{(28b)}\\
&= \frac{\Delta^2}{8}\hat{\mathbf{i}} + \frac{\Delta^2}{4}\hat{\mathbf{j}} + \frac{\Delta^2}{4}\hat{\mathbf{k}} && \text{(28c)}
\end{aligned}$$

We could have also used the following (check it!)

$$\overrightarrow{\mathbf{A}}_{ABC} = \frac{1}{2}\overrightarrow{\mathbf{CA}} \times \overrightarrow{\mathbf{CB}} = \frac{1}{2}\overrightarrow{\mathbf{BC}} \times \overrightarrow{\mathbf{BA}} = \frac{\Delta^2}{8}\hat{\mathbf{i}} + \frac{\Delta^2}{4}\hat{\mathbf{j}} + \frac{\Delta^2}{4}\hat{\mathbf{k}} \tag{29}$$

Note that the absolute value of $\overrightarrow{\mathbf{A}}_{ABC}$ gives same answer as obtained by Eq. (24)

$$\left\|\overrightarrow{\mathbf{A}}_{ABC}\right\| = \sqrt{\overrightarrow{\mathbf{A}}_{ABC} \cdot \overrightarrow{\mathbf{A}}_{ABC}} = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{30a}$$

$$= \sqrt{\left(\frac{\Delta^2}{8}\right)^2 + \left(\frac{\Delta^2}{4}\right)^2 + \left(\frac{\Delta^2}{4}\right)^2} \tag{30b}$$

$$= \frac{3}{8}\Delta^2 \tag{30c}$$

The unit normal to face ABC is found using Eqs. (28) and (30)

$$\hat{\mathbf{n}}_{ABC} = \frac{\overrightarrow{\mathbf{A}}_{ABC}}{\left\|\overrightarrow{\mathbf{A}}_{ABC}\right\|} = \frac{\frac{\Delta^2}{8}\hat{\mathbf{i}} + \frac{\Delta^2}{4}\hat{\mathbf{j}} + \frac{\Delta^2}{4}\hat{\mathbf{k}}}{\frac{3}{8}\Delta^2} \tag{31a}$$

$$= \frac{1}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{2}{3}\hat{\mathbf{k}} \tag{31b}$$

Stress vector is found as follows

$$\overrightarrow{\mathbf{T}}^{ABC} = [\boldsymbol{\sigma}] \cdot \hat{\mathbf{n}}_{ABC} \tag{32a}$$

$$= \begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \cdot \begin{Bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{Bmatrix} = \begin{Bmatrix} 40 \\ 20/3 \\ -40/3 \end{Bmatrix} \text{ MPa} \tag{32b}$$

$$= 40\,\hat{\mathbf{i}} + \frac{20}{3}\,\hat{\mathbf{j}} - \frac{40}{3}\,\hat{\mathbf{k}} \text{ MPa} \tag{32c}$$

$$= 40\,\hat{\mathbf{i}} + 6.667\,\hat{\mathbf{j}} - 13.333\,\hat{\mathbf{k}} \text{ MPa} \tag{32d}$$