AOE 3024: Thin Walled Structures
Solution to Homework # 2

NAME	**PLEDGE**

The state of stress at a point in a component is given as

$$\begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \text{ MPa} \tag{1}$$

a. Determine the normal and tangential components of the stress vector acting on the face ABC. Note that $OA = 2OB = 2OC = \Delta$ meters (5 points).

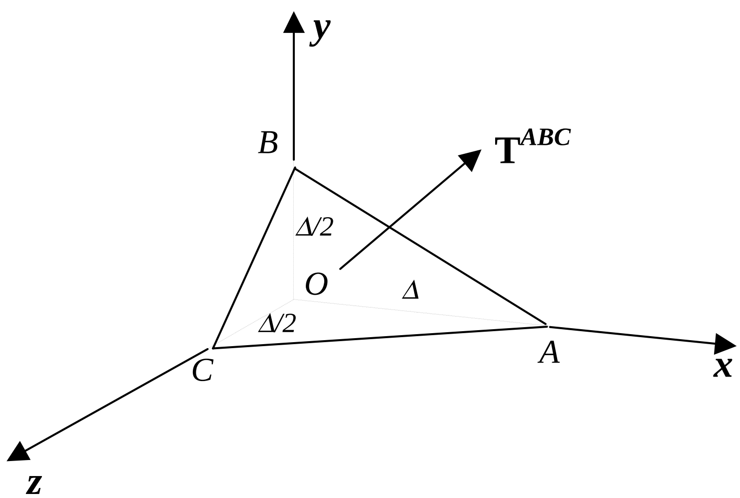

Fig. 1 The stress vector acting on the face ABC from homework # 1.

From homework # 1, the area of triangle ABC is

$$\overrightarrow{\mathbf{A}}_{ABC} = \frac{1}{2}\,\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}} = \frac{1}{2}\,\overrightarrow{\mathbf{CA}} \times \overrightarrow{\mathbf{CB}} = \frac{1}{2}\,\overrightarrow{\mathbf{BC}} \times \overrightarrow{\mathbf{BA}} = \frac{\Delta^2}{8}\hat{\mathbf{i}} + \frac{\Delta^2}{4}\hat{\mathbf{j}} + \frac{\Delta^2}{4}\hat{\mathbf{k}} \tag{2}$$

The unit normal to face ABC is

$$\hat{\mathbf{n}}_{ABC} = \frac{\overrightarrow{\mathbf{A}}_{ABC}}{\|\overrightarrow{\mathbf{A}}_{ABC}\|} = \frac{1}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{2}{3}\hat{\mathbf{k}} \tag{3}$$

Stress vector and magnitude is

$$\overrightarrow{\mathbf{T}}^{ABC} = [\boldsymbol{\sigma}] \cdot \hat{\mathbf{n}}_{ABC} = 40\,\hat{\mathbf{i}} + \frac{20}{3}\,\hat{\mathbf{j}} - \frac{40}{3}\,\hat{\mathbf{k}} \text{ MPa} \tag{4a}$$

$$\|\overrightarrow{\mathbf{T}}^{ABC}\| = \frac{20}{3}\sqrt{41} \text{ MPa} \tag{4b}$$

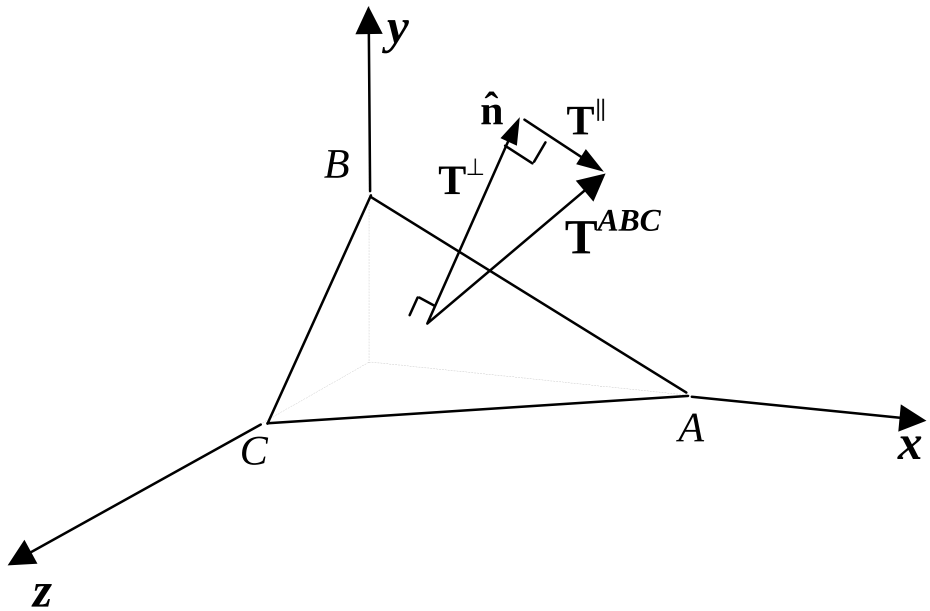

Fig. 2 Figure shows vector normal $\mathbf{T}^{\parallel}$ and vector tangential $\mathbf{T}^{\perp}$ to surface ABC

Now the normal (or orthogonal) vector $\overrightarrow{\mathbf{T}}^{\perp}$ is the projection of $\overrightarrow{\mathbf{T}}^{ABC}$ onto the a line through the unit normal $\hat{\mathbf{n}}_{ABC}$. The magnitude of $\overrightarrow{\mathbf{T}}^{\perp}$ is found using the inner (dot) product:

$$\|\overrightarrow{\mathbf{T}}^{\perp}\| = \|\overrightarrow{\mathbf{T}}^{ABC}\|\|\hat{\mathbf{n}}_{ABC}\|\cos\theta \quad \text{where } \theta \text{ is the angle between vectors } \overrightarrow{\mathbf{T}}^{\perp} \text{ and } \overrightarrow{\mathbf{T}}^{ABC} \tag{5a}$$

$$= \overrightarrow{\mathbf{T}}^{ABC} \cdot \hat{\mathbf{n}}_{ABC} \tag{5b}$$

$$= T_x^{ABC}\, n_x + T_y^{ABC}\, n_y + T_z^{ABC}\, n_z \tag{5c}$$

$$= (40)\left(\frac{1}{3}\right) + \left(\frac{20}{3}\right)\left(\frac{2}{3}\right) + \left(-\frac{40}{3}\right)\left(\frac{2}{3}\right) \text{ MPa} \tag{5d}$$

$$= \frac{80}{9} \text{ MPa} = 8.88889 \text{ MPa} \tag{5e}$$

The vector projection can be found by multiplying the magnitude of $\overrightarrow{\mathbf{T}}^{\perp}$ with the unit normal in that direction (in this case is $\hat{\mathbf{n}}_{ABC}$)

$$\overrightarrow{\mathbf{T}}^{\perp} = \|\overrightarrow{\mathbf{T}}^{\perp}\|\hat{\mathbf{n}}_{ABC} \tag{6a}$$

$$= \frac{80}{9}\left(\frac{1}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{2}{3}\hat{\mathbf{k}}\right) \text{ MPa} \tag{6b}$$

$$= \frac{80}{27}\,\hat{\mathbf{i}} + \frac{160}{27}\,\hat{\mathbf{j}} + \frac{160}{27}\,\hat{\mathbf{k}} \text{ MPa} \tag{6c}$$

$$= 2.96296\,\hat{\mathbf{i}} + 5.92593\,\hat{\mathbf{j}} + 5.92593\,\hat{\mathbf{k}} \text{ MPa} \tag{6d}$$

The tangential (or parallel) vector $\overrightarrow{\mathbf{T}}^{\parallel}$ is orthogonal to unit normal vector $\hat{\mathbf{n}}_{ABC}$. Now using the fact that

$$\overrightarrow{\mathbf{T}}^{\parallel} + \overrightarrow{\mathbf{T}}^{\perp} = \overrightarrow{\mathbf{T}}^{ABC} \tag{7}$$

we get

$$\overrightarrow{\mathbf{T}}^{\parallel} = \overrightarrow{\mathbf{T}}^{ABC} - \overrightarrow{\mathbf{T}}^{\perp} \tag{8a}$$

$$= \left(40\,\hat{\mathbf{i}} + \frac{20}{3}\,\hat{\mathbf{j}} - \frac{40}{3}\,\hat{\mathbf{k}}\right) - \left(\frac{80}{27}\,\hat{\mathbf{i}} + \frac{160}{27}\,\hat{\mathbf{j}} + \frac{160}{27}\,\hat{\mathbf{k}}\right)\ \text{MPa} \tag{8b}$$

$$= \frac{1000}{27}\,\hat{\mathbf{i}} + \frac{20}{27}\,\hat{\mathbf{j}} - \frac{520}{27}\,\hat{\mathbf{k}}\ \text{MPa} \tag{8c}$$

$$= 37.037\,\hat{\mathbf{i}} + 0.740741\,\hat{\mathbf{j}} - 19.2593\,\hat{\mathbf{k}}\ \text{MPa} \tag{8d}$$

The magnitude of $\overrightarrow{\mathbf{T}}^{\parallel}$ is found taking the absolute value,

$$\|\overrightarrow{\mathbf{T}}^{\parallel}\| = \sqrt{\overrightarrow{\mathbf{T}}^{\parallel} \cdot \overrightarrow{\mathbf{T}}^{\parallel}} \tag{9a}$$

$$= \sqrt{T_x^{\parallel}\,T_x^{\parallel} + T_y^{\parallel}\,T_y^{\parallel} + T_z^{\parallel}\,T_z^{\parallel}} \tag{9b}$$

$$= \sqrt{\left(\frac{1000}{27}\right)^2 + \left(\frac{20}{27}\right)^2 + \left(-\frac{520}{27}\right)^2}\ \text{MPa} \tag{9c}$$

$$= \frac{20}{9}\sqrt{353}\ \text{MPa} = 41.7518\ \text{MPa} \tag{9d}$$

We could have also calculated the magnitude of $\overrightarrow{\mathbf{T}}^{\parallel}$ using pythagoras,

$$\|\overrightarrow{\mathbf{T}}^{\parallel}\|^2 + \|\overrightarrow{\mathbf{T}}^{\perp}\|^2 = \|\overrightarrow{\mathbf{T}}^{ABC}\|^2 \tag{10}$$

Therefore,

$$\|\overrightarrow{\mathbf{T}}^{\parallel}\| = \sqrt{\|\overrightarrow{\mathbf{T}}^{ABC}\|^2 - \|\overrightarrow{\mathbf{T}}^{\perp}\|^2} \tag{11a}$$

$$\|\overrightarrow{\mathbf{T}}^{\parallel}\| = \sqrt{\left(\frac{20}{3}\sqrt{41}\right)^2 - \left(\frac{80}{9}\right)^2} \tag{11b}$$

$$= \frac{20}{9}\sqrt{353}\ \text{MPa} = 41.7518\ \text{MPa} \tag{11c}$$

b. Determine the three Principal stresses and corresponding Principal planes. (20 points)

For a principal plane, Cauchy's equations can be written in matrix form as follows

$$\begin{bmatrix} \sigma_{xx}-\lambda & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\lambda & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{12a}$$

$$\begin{bmatrix} 40-\lambda & 40 & 0 \\ 40 & 50-\lambda & -60 \\ 0 & -60 & 40-\lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{12b}$$

These equations have the trivial solution $n_x = n_y = n_z = 0$, which is not allowed, since n_x, n_y, and n_z are the components of a unit vector, satisfying

$$n_x^2 + n_y^2 + n_z^2 = 1 \tag{13}$$

Equations in (12) possess a nontrivial solution if the three equations are not independent of each other. In other words, the determinant of the matrix of coefficients of n_x, n_y, and n_z must vanish:

$$det \begin{bmatrix} \sigma_{xx}-\lambda & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\lambda & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\lambda \end{bmatrix} = \begin{vmatrix} 40-\lambda & 40 & 0 \\ 40 & 50-\lambda & -60 \\ 0 & -60 & 40-\lambda \end{vmatrix} = 0 \tag{14}$$

The characteristic equation obtained by expanding the determinant can be expressed in terms of the stress invariants as follows

$$\lambda^3 - I_1\,\lambda^2 + I_2\,\lambda - I_3 = 0 \tag{15}$$

where I_i's are the stress invariants. Obtained by directly expanding the characteristic equation

$$\begin{vmatrix} 40-\lambda & 40 & 0 \\ 40 & 50-\lambda & -60 \\ 0 & -60 & 40-\lambda \end{vmatrix} = 0 \tag{16}$$

$$(40-\lambda)\begin{vmatrix} 50-\lambda & -60 \\ -60 & 40-\lambda \end{vmatrix} - (40)\begin{vmatrix} 40 & -60 \\ 0 & 40-\lambda \end{vmatrix} + (0)\begin{vmatrix} 40 & 50-\lambda \\ 0 & -60 \end{vmatrix} = 0$$

$$\lambda^3 - 130\,\lambda^2 + 400\,\lambda + 128000 = 0 \tag{17}$$

from which I_1, I_2, and I_3 can be obtained by comparing to Eq. (15). The three roots of the characteristic equation, Eq. (15), are the principle stresses and can be obtained analytically: (note that to use the following equations we must work in **radians**)

$$\sigma_1 = \frac{I_1}{3} + \frac{2}{3}\sqrt{I_1^2 - 3\,I_2}\,\cos\left(\frac{\alpha}{3}\right) \tag{18}$$

$$\sigma_2 = \frac{I_1}{3} + \frac{2}{3}\sqrt{I_1^2 - 3\,I_2}\,\cos\left(\frac{\alpha}{3} + \frac{2\,\pi}{3}\right) \tag{19}$$

$$\sigma_3 = \frac{I_1}{3} + \frac{2}{3}\sqrt{I_1^2 - 3\,I_2}\,\cos\left(\frac{\alpha}{3} + \frac{4\,\pi}{3}\right) \tag{20}$$

$$\alpha = \cos^{-1}\left[\frac{2I_1^3 - 9I_1I_2 + 27I_3}{2\sqrt{(I_1^2 - 3I_2)^3}}\right] \tag{21}$$

$$\alpha = \cos^{-1}\left[\frac{235}{157\,\sqrt{157}}\right] = 1.45105\,\text{rads (keep in radians)} \tag{22}$$

$$\sigma_1 = 117.284 \text{ MPa} \qquad \sigma_2 = -27.2842 \text{ MPa} \qquad \sigma_3 = 40.00 \text{ MPa} \tag{23}$$

Principal Plane: $\mathbf{n}^{(1)}$

To find $\mathbf{n}^{(1)}$, the principle direction of $\sigma_1 = 117.284$ MPa, we substitute $\lambda = \sigma_1$ into any two equations of Eq. (12) (but not all three). This will give two of the three components of $\mathbf{n}^{(1)}$ ($n_x^{(1)}$, $n_y^{(1)}$, and $n_z^{(1)}$) and the last component is obtained with

$$n_x^2 + n_y^2 + n_z^2 = 1$$

Therefore,

$$\begin{bmatrix} -77.2842 & 40 & 0 \\ 40 & -67.2842 & -60 \\ 0 & -60 & -77.2842 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{24a}$$

$$\begin{aligned} -77.2842\, n_x^{(1)} &+ 40\, n_y^{(1)} &+ 0 &= 0 \\ 40\, n_x^{(1)} &+ -67.2842\, n_y^{(1)} &+ -60\, n_z^{(1)} &= 0 \\ 0 &+ -60\, n_y^{(1)} &+ -77.2842\, n_z^{(1)} &= 0 \end{aligned} \tag{24b}$$

Using the first two equations (we could have used any two equations) and solving all variables in terms of $n_z^{(1)}$ (we could have solved in terms of any other component)

$$n_x^{(1)} = -0.666667\, n_z^{(1)} \qquad n_y^{(1)} = -1.28807\, n_z^{(1)} \tag{25}$$

Now obtain $n_z^{(1)}$ using Eq. (13)

$$n_x^2 + n_y^2 + n_z^2 = 1 \tag{26a}$$

$$\left(-0.666667\, n_z^{(1)}\right)^2 + \left(-1.28807\, n_z^{(1)}\right)^2 + \left(n_z^{(1)}\right)^2 = 1 \tag{26b}$$

$$3.10357 \left(n_z^{(1)}\right)^2 = 1 \tag{26c}$$

Taking the positive sign (arbitrarily) of $n_z^{(1)}$:

$$n_z^{(1)} = 0.567635 \qquad n_x^{(1)} = -0.378424 \qquad n_y^{(1)} = -0.731154 \tag{27}$$

Principal Plane: $\mathbf{n^{(2)}}$

To find $\mathbf{n}^{(2)}$, the principle direction of $\sigma_2 = -27.2842$ MPa, we substitute $\lambda = \sigma_2$ into any two equations of Eq. (12) (but not all three). This will give two of the three components of $\mathbf{n}^{(2)}$ ($n_x^{(2)}$, $n_y^{(2)}$, and $n_z^{(2)}$) and the last component is obtained with

$$n_x^2 + n_y^2 + n_z^2 = 1$$

Therefore,

$$\begin{bmatrix} 67.2842 & 40 & 0 \\ 40 & 77.2842 & -60 \\ 0 & -60 & 67.2842 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{28a}$$

$$\begin{aligned} 67.2842\, n_x^{(2)} \;&+\; 40\, n_y^{(2)} \;+\; 0 &= 0 \\ 40\, n_x^{(2)} \;&+\; 77.2842\, n_y^{(2)} \;+\; -60\, n_z^{(2)} &= 0 \\ 0 \;&+\; -60\, n_y^{(2)} \;+\; 67.2842\, n_z^{(2)} &= 0 \end{aligned} \tag{28b}$$

Using the first two equations (we could have used any two equations) and solving all variables in terms of $n_z^{(2)}$ (we could have solved in terms of any other component)

$$n_x^{(2)} = -0.666667\, n_z^{(2)} \qquad n_y^{(2)} = 1.1214\, n_z^{(2)} \tag{29}$$

Now obtain $n_z^{(2)}$ using Eq. (13)

$$n_x^2 + n_y^2 + n_z^2 = 1 \tag{30a}$$

$$\left(-0.666667\, n_z^{(2)}\right)^2 + \left(1.1214\, n_z^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 = 1 \tag{30b}$$

$$2.70199 \left(n_z^{(2)}\right)^2 = 1 \tag{30c}$$

Taking the positive sign (arbitrarily) of $n_z^{(2)}$:

$$n_z^{(2)} = 0.608357 \qquad n_x^{(2)} = -0.405571 \qquad n_y^{(2)} = 0.682213 \tag{31}$$

Principal Plane: $\mathbf{n^{(3)}}$

To find $\mathbf{n^{(3)}}$, the principle direction of $\sigma_3 = 40.00$ MPa, we substitute $\lambda = \sigma_3$ into any two equations of Eq. (12) (but not all three). This will give two of the three components of $\mathbf{n^{(3)}}$ ($n_x^{(3)}$, $n_y^{(3)}$, and $n_z^{(3)}$) and the last component is obtained with

$$n_x^2 + n_y^2 + n_z^2 = 1$$

Therefore,

$$\begin{bmatrix} 0 & 40 & 0 \\ 40 & 10 & -60 \\ 0 & -60 & 0 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}^{(3)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{32a}$$

$$\begin{aligned} 0\,n_x^{(3)} &+ 40\,n_y^{(3)} &+ 0 &= 0 \\ 40\,n_x^{(3)} &+ 10\,n_y^{(3)} &+ -60\,n_z^{(3)} &= 0 \\ 0 &+ -60\,n_y^{(3)} &+ 0 &= 0 \end{aligned} \tag{32b}$$

Using the first two equations (we could have used any two equations) and solving all variables in terms of $n_z^{(3)}$ (we could have solved in terms of any other component)

$$n_x^{(3)} = 1.5\,n_z^{(3)} \qquad n_y^{(3)} = 0\,n_z^{(3)} \tag{33}$$

Now obtain $n_z^{(3)}$ using Eq. (13)

$$n_x^2 + n_y^2 + n_z^2 = 1 \tag{34a}$$

$$\left(1.5\,n_z^{(3)}\right)^2 + \left(0\,n_z^{(3)}\right)^2 + \left(n_z^{(3)}\right)^2 = 1 \tag{34b}$$

$$3.25\left(n_z^{(3)}\right)^2 = 1 \tag{34c}$$

Taking the positive sign (arbitrarily) of $n_z^{(3)}$:

$$n_z^{(3)} = 0.5547 \qquad n_x^{(3)} = 0.83205 \qquad n_y^{(3)} = 0 \tag{35}$$