AOE 3024: Thin Walled Structures
Solutions to Homework # 4

The state of stress at a point in a component is given as

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa} \tag{1}$$

a) Determine the factor of safety using Tresca (Maximum Shear stress) and Von-Mises failure criteria. Assume the yield stress to be $\sigma_y = 250$ MPa

Factor of safety can be defined as the ratio of the allowable strength to the required strength

$$FS = \frac{\text{Allowable stresses}}{\text{Required stresses}} = \frac{\sigma_{all}}{\sigma_{req}} \tag{2}$$

and is always greater than one.

First, we calculate the principal stresses.

In general, when all three principal stresses are needed the eigenvalue approach is easier (See Homework # 2 for details). However, for problems where $\tau_{xz} = \tau_{yz} = \tau_{zx} = \tau_{zy} = 0$, one of the principal stresses will be $\sigma_3 = \sigma_{zz}$. Moreover, for a plane stress problem $\sigma_3 = \sigma_{zz} = 0$. Therefore, Mohr's Circle for plane stress problems can be used.

Using the Mohr's Circle

$$\sigma_{ave} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{(40) + (50)}{2} \text{ MPa} = 45 \text{ MPa} \tag{3}$$

$$\sigma_{dif} = \frac{\sigma_{xx} - \sigma_{yy}}{2} = \frac{(40) - (50)}{2} \text{ MPa} = -5 \text{ MPa} \tag{4}$$

$$R = \sqrt{\tau_{xy}^2 + \sigma_{dif}^2} = \sqrt{(40)^2 + (-5)^2} \text{ MPa} = 40.3113 \text{ MPa} \tag{5}$$

$$\sigma_1 = \sigma_{ave} + R = (45) + (40.3113) = 85.3113 \text{ MPa} \tag{6}$$

$$\sigma_2 = \sigma_{ave} - R = (45) - (40.3113) = 4.6887 \text{ MPa} \tag{7}$$

$$\sigma_3 = 0 \text{ MPa (for plane stress)} \tag{8}$$

Using the eigenvalue approach

Cauchy's equations can be written in matrix form as follows

$$\begin{bmatrix} 40-\lambda & 40 & 0 \\ 40 & 50-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{9}$$

These equations possess a nontrivial solution when the determinant of the matrix of coefficients of n_x, n_y, and n_z vanishes. This determinant leads to the characteristic equation

$$\lambda^3 - 90\,\lambda^2 + 400\,\lambda = \left(\lambda^2 - 90\,\lambda + 400\right)\lambda = 0 \tag{10}$$

The three roots of the above characteristic equation are the principle stresses:

$$\sigma_1 = 85.3113 \text{ MPa} \qquad \sigma_2 = 4.6887 \text{ MPa} \qquad \sigma_3 = 0.0 \text{ MPa} \tag{11}$$

I. Tresca (Maximum Shear Stress) Failure Criterion

Tresca criterion is a method to predict yielding in 3-D state of stress and is defined as:

$$\sigma_{max} - \sigma_{min} = \pm\sigma_y \tag{12}$$

where

$$\sigma_{max} = \max[\sigma_1, \sigma_2, \sigma_3] = \max[85.3113, 4.68871, 0] = 85.3113 \text{ MPa} \tag{13}$$

$$\sigma_{min} = \min[\sigma_1, \sigma_2, \sigma_3] = \min[85.3113, 4.68871, 0] = 0 \text{ MPa} \tag{14}$$

The required strength is the overall maximum shear stress and is defined as

$$\sigma_{req} = \tau_{max}\Big|_{3-D} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{85.3113 - 0}{2} \text{ MPa} = 42.6556 \text{ MPa} \tag{15}$$

The cross section of a uniaxial test specimen is a principal plane, and the normal stress on that plane is the only nonzero principal stress. Therefore, the allowable strength is the maximum shear stress in a one dimensional tension test and is defined as

$$\sigma_{all} = \tau_{max}\Big|_{1-D} = \frac{\sigma_y}{2} = \frac{250}{2} \text{ MPa} = 125 \text{ MPa} \tag{16}$$

Then the factor of safety is

$$FS = \frac{\sigma_{all}}{\sigma_{req}} = \frac{125}{42.6556} = 2.93044 \tag{17}$$

II. Von Mises Failure Criterion

Von Mises stress is another method to predict yielding in 3-D state of state and can be defined in different, but equivalent, ways

$$\sigma_M = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}} \tag{18a}$$

$$\sigma_M = \sqrt{\frac{1}{2}\left[(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{xx}-\sigma_{zz})^2+(\sigma_{yy}-\sigma_{zz})^2\right]+3\left(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2\right)} \tag{18b}$$

For plane stress problems ($\tau_{yz}=\tau_{xz}=\sigma_{zz}=0$), for which $\sigma_3=\sigma_{zz}=0$, the above simplifies to

$$\sigma_M = \sqrt{\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2} \tag{19a}$$

$$\sigma_M = \sqrt{\sigma_{xx}^2+\sigma_{yy}^2-\sigma_{xx}\sigma_{yy}+3\,\tau_{xy}^2} \tag{19b}$$

Therefore, the required strength is the von Mises stress. Using equation for plane stress

$$\sigma_{req}=\sigma_M=\sqrt{(85.3113)^2-(85.3113)(4.68871)+(4.68871)^2}\text{ MPa}=83.0662\text{ MPa} \tag{20}$$

The allowable strength is the yield stress

$$\sigma_{all}=\sigma_y=250\text{ MPa} \tag{21}$$

Then the factor of safety is

$$FS=\frac{\sigma_{all}}{\sigma_{req}}=\frac{250}{83.0662}=3.00965 \tag{22}$$

Note that: (1) the advantage in using von Mises criteria is that one does not need to find the principal stresses in order to calculate σ_{M}; (2) the lower factor of safety predicted by the maximum shear stress criterion shows it is slightly conservative with respect to von Mises prediction.

b) Your solution should clearly indicate the given state of stress in the σ_a-σ_b failure diagram.

First, note that we will nondimensionalize all stresses by dividing them by the yield stress. Therefore σ_a and σ_b will be nondimensional quantities

$$\sigma_a=\frac{\sigma_1}{\sigma_y} \qquad \sigma_b=\frac{\sigma_2}{\sigma_y} \tag{23}$$

(Nondimensionalization is not necessary. However, it helps us better understand the different criterions.)

Therefore, let the location of the present state of stress be $\sigma_{a_1}, \sigma_{b_1}$

$$\sigma_{a_1} = \frac{85.3113}{250} = 0.34124 \qquad \sigma_{b_1} = \frac{4.68871}{250} = 0.01875 \tag{24}$$

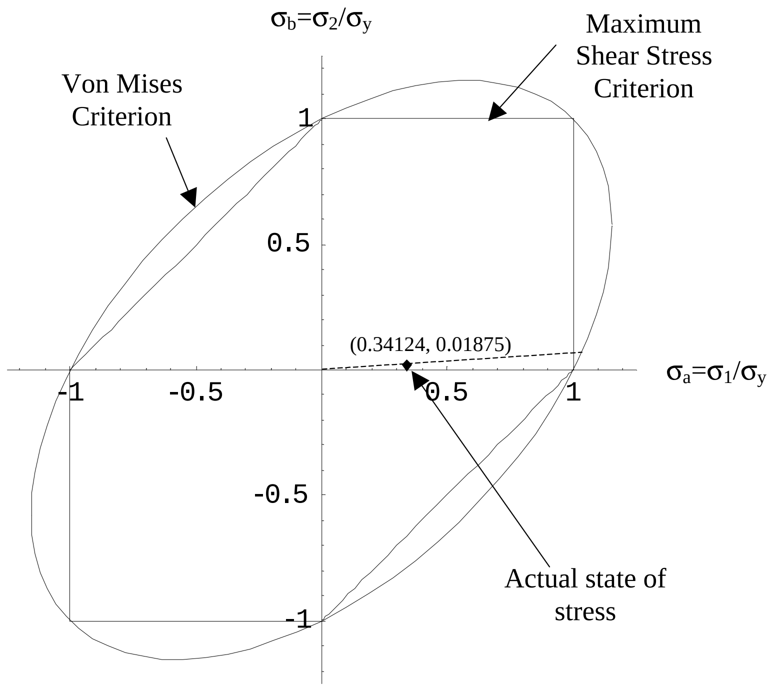

Fig. 1 Location of the stress state of the present problem

Note that the factor of safety can be also found from the σ_a-σ_b failure diagram

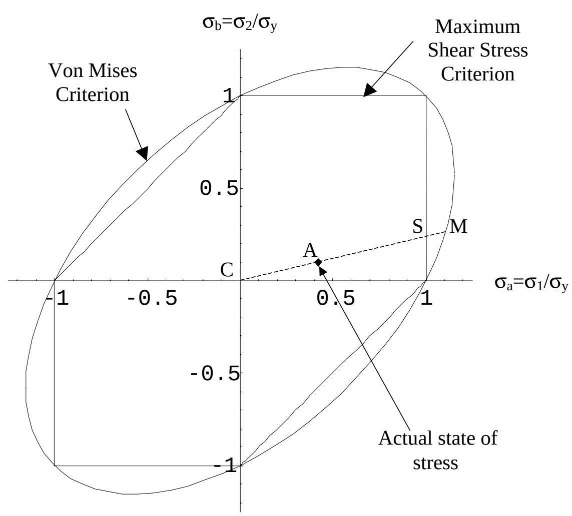

Fig. 2 General location of the stress state

For von Mises Failure Criterion: Then the factor of safety is

$$FS = \frac{\sigma_{all}}{\sigma_{req}} = \frac{\overline{CM}}{\overline{CA}} = 3.00965 \tag{25}$$

For Tresca Criterion: Then the factor of safety is

$$FS = \frac{\sigma_{all}}{\sigma_{req}} = \frac{\overline{CS}}{\overline{CA}} = 2.93044 \tag{26}$$