# AOE 3024: Thin Walled Structures
# Solutions to Homework # 5

**PROBLEM 1. The three displacement components $u$, $v$, and $w$ for a cantilever beam of length $l$, under a tip load $P$ are given as:**

$$u = u(x,y,z) = -\frac{P\,x^2\,y}{2\,E\,I} - \nu\,\frac{P\,y^3}{6\,E\,I} + \frac{P\,y^3}{6\,G\,I} + \left(\frac{P\,l^2}{2\,E\,I} - \frac{P\,c^2}{2\,G\,I}\right)y \tag{1}$$

$$v = v(x,y,z) = \nu\,\frac{P\,x\,y^2}{2\,E\,I} + \frac{P\,x^3}{6\,E\,I} - \frac{P\,l^2\,x}{2\,E\,I} + \frac{P\,l^3}{3\,E\,I} \tag{2}$$

$$w = w(x,y,z) = 0 \tag{3}$$

**Determine all strain components.**
Recall that for equilibrium

$$[\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \epsilon_{zz} \end{bmatrix} \tag{4}$$

Note that all derivatives of $w$ vanish because $w = 0$. Also, note that $u$ and $v$ are not functions of $z$: $u(x,y,z) = u(x,y)$ and $v(x,y,z) = v(x,y)$. Therefore, all derivatives respect to $z$ vanish

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = 0 \tag{5}$$

$$\gamma_{yz} = 2\,\epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \tag{6}$$

$$\gamma_{xz} = 2\,\epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \tag{7}$$

Determining the nonzero of the strains:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = -\frac{P\,x\,y}{E\,I} \tag{8}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = \nu\,\frac{P\,x\,y}{E\,I} \tag{9}$$

$$\gamma_{xy} = 2\,\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{c^2\,P}{2\,G\,I} + \frac{P\,y^2}{2\,G\,I} \tag{10}$$

For isotropic materials, $G = E/[2(1+\nu)]$ therefore,

$$\gamma_{xy} = -\frac{c^2\,P}{E\,I} + \frac{P\,y^2}{E\,I} - \frac{c^2\,P\,\nu}{E\,I} + \frac{P\,y^2\,\nu}{E\,I} = \frac{P\,(y-c)\,(y+c)\,(1+\nu)}{E\,I}$$

These results are identical to those obtained by T. H. G. Megson (course textbook Aircraft Structures for engineering students, 3rd ed., page 45) The only difference is that Megson's takes the height of the beam as $b$ instead of $2\,c$. In other words, our $c$ is equal to $b/2$ in Megson's textbook.

**PROBLEM 2. The strain gage measurements from a rosette are given as:**

$$\epsilon_{xx} = 2000\mu \tag{11}$$
$$\epsilon_{xx+45^\circ} = 2000\mu \tag{12}$$
$$\epsilon_{yy} = 950\mu \tag{13}$$

**Determine the principal strains from the above components.**

**Step I. Need to calculate all strains components**

In order to calculate the principal strains, we need the actual strains for the above measurements. These can be obtained by solving the following system of equations:

$$\epsilon_1 = \epsilon_{xx}\cos^2\theta_1 + \epsilon_{yy}\sin^2\theta_1 + \gamma_{xy}\sin\theta_1\cos\theta_1 \tag{14}$$

$$\epsilon_2 = \epsilon_{xx}\cos^2\theta_2 + \epsilon_{yy}\sin^2\theta_2 + \gamma_{xy}\sin\theta_2\cos\theta_2 \tag{15}$$

$$\epsilon_3 = \epsilon_{xx}\cos^2\theta_3 + \epsilon_{yy}\sin^2\theta_3 + \gamma_{xy}\sin\theta_3\cos\theta_3 \tag{16}$$

where the $\theta_i$'s are measured counterclockwise from the $x$-axis. The above can also be written in matrix form,

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_1 & \sin^2\theta_1 & \sin\theta_1\cos\theta_1 \\ \cos^2\theta_2 & \sin^2\theta_2 & \sin\theta_2\cos\theta_2 \\ \cos^2\theta_3 & \sin^2\theta_3 & \sin\theta_3\cos\theta_3 \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{17}$$

Note that

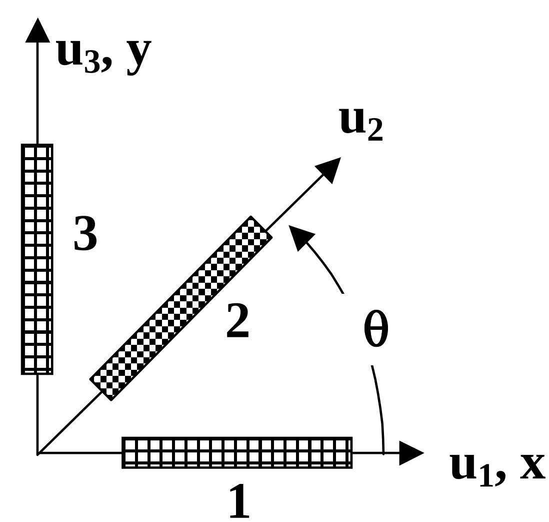


**Fig. 1 3-gage rosette**

$$\theta_1 = 0^\circ, \qquad \theta_2 = 45^\circ, \qquad \theta_3 = 90^\circ \tag{18}$$
$$\epsilon_1 = \epsilon_{xx} = 2000\mu, \qquad \epsilon_2 = \epsilon_{xx+45^\circ} = 2000\mu, \qquad \epsilon_3 = \epsilon_{yy} = 950\mu \tag{19}$$

Therefore,

$$\left\{\begin{array}{c}2000\\2000\\950\end{array}\right\}\mu \quad = \quad \left[\begin{array}{ccc}1 & 0 & 0\\1/2 & 1/2 & 1/2\\0 & 1 & 0\end{array}\right]\left\{\begin{array}{c}\epsilon_{xx}\\\epsilon_{yy}\\\gamma_{xy}\end{array}\right\} \tag{20a}$$

$$\left\{\begin{array}{c}\epsilon_{xx}\\\epsilon_{yy}\\\gamma_{xy}\end{array}\right\} \quad = \quad \left[\begin{array}{ccc}1 & 0 & 0\\1/2 & 1/2 & 1/2\\0 & 1 & 0\end{array}\right]^{-1}\left\{\begin{array}{c}2000\\2000\\950\end{array}\right\}\mu = \left\{\begin{array}{c}2000\\950\\1050\end{array}\right\}\mu \tag{20b}$$

Note that for this rosette $\epsilon_1 = \epsilon_{xx}$ and $\epsilon_3 = \epsilon_{yy}$ and the only unknown is the shear strain $\gamma_{xy}$, which could have been directly calculated using the transformation relationship for $\epsilon_2$:

$$\epsilon_2 = \epsilon(\theta) \quad = \quad \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\,\cos\theta \tag{21}$$

$$= \quad \epsilon_{ave} + \epsilon_{dif}\,\cos 2\theta + \frac{\gamma_{xy}}{2}\,\sin 2\theta$$

$$\epsilon(45^\circ) \quad = \quad \epsilon_{ave} + \frac{\gamma_{xy}}{2} \tag{22}$$

where,

$$\epsilon_{ave} \quad = \quad \frac{\epsilon_{xx} + \epsilon_{yy}}{2} = \frac{(2000) + (950)}{2}\,\mu = 1475\,\mu \tag{23}$$

$$\epsilon_{dif} \quad = \quad \frac{\epsilon_{xx} - \epsilon_{yy}}{2} = \frac{(2000) - (950)}{2}\,\mu = 525\,\mu \tag{24}$$

Now, half the shear strain, $\gamma_{xy}/2$, is

$$\frac{\gamma_{xy}}{2} \quad = \quad \epsilon(45^\circ) - \epsilon_{ave} \tag{25}$$

$$\gamma_{xy} \quad = \quad 2\,\epsilon(45^\circ) - 2\,\epsilon_{ave} \tag{26}$$

$$= \quad 2\,(2000\,\mu) - 2\,(1475\,\mu) = 1050\,\mu$$

$$\frac{\gamma_{xy}}{2} \quad = \quad 525\,\mu \tag{27}$$

**Step II. Calculate the radius and Mohr's circle center**

$$R = \sqrt{\left(\frac{\gamma_{xy}}{2}\right)^2 + \epsilon_{dif}^2} = \sqrt{(525)^2 + (525)^2}\,\mu = 742.462\,\mu \quad (28)$$

$$C = C(\epsilon_{ave}, 0) = C(1475\,\mu, 0) \quad (29)$$

**Step III. Draw the circle and locate all points**

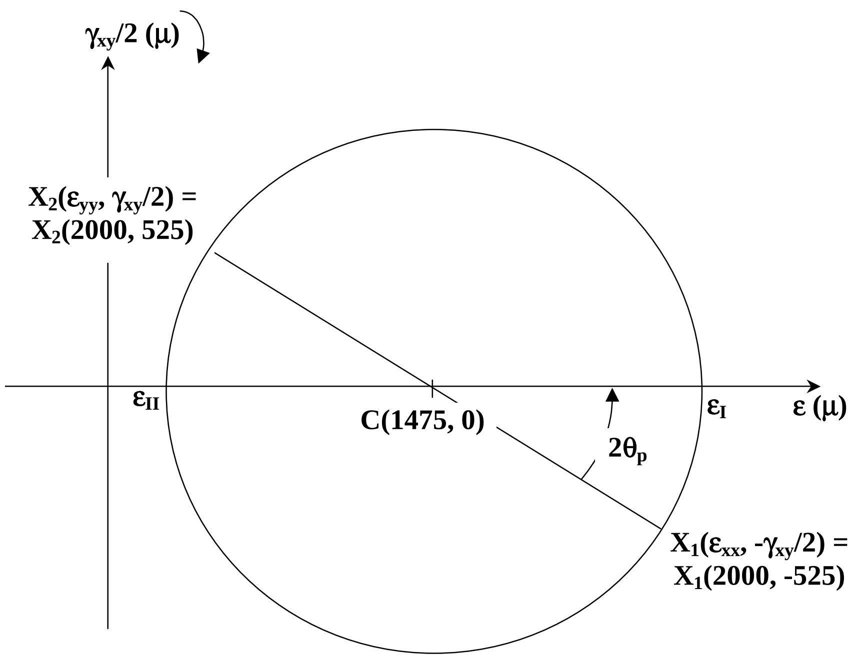


**Fig. 2 Mohr's circle for plane strain in the $xy$ plane**

**Step IV. Calculate angles: (All angles measured counterclockwise)**

Principal strains act at a rotation angle $\theta_p$

$$2\theta_p = \tan^{-1}\left[\frac{\gamma_{xy}/2}{\sigma_{dif}}\right] = \tan^{-1}\left[\frac{(525)}{(525)}\right] = 45^\circ \quad \Rightarrow \quad \theta_p = 22.5^\circ \quad (30)$$

Note $2\,\theta_p$ is measured from $\overline{X_1C}$ to positive $\epsilon$-axis. Maximum shear strain acts on an element rotated at an angle $\theta_s$

$$2\theta_s = 2\theta_p \pm 90^\circ = 45^\circ \pm 90^\circ \quad (31a)$$

$$\theta_s = \theta_p \pm 45^\circ = 22.5^\circ \pm 45^\circ \quad (31b)$$

**Step V. Determine the principal strains**

$$\epsilon_I = \epsilon_{ave} + R = (1475\,\mu) + (742.462\,\mu) = 2217.46\,\mu \quad (32)$$

$$\epsilon_{II} = \epsilon_{ave} - R = (1475\,\mu) - (742.462\,\mu) = 732.538\,\mu \quad (33)$$

**Step VI. Determine the maximum shear strain**

$$\left.\frac{\gamma_{max}}{2}\right|_{in-plane} = R = \frac{\epsilon_I - \epsilon_{II}}{2} = 742.462\,\mu \quad (34)$$

$$\left.\gamma_{max}\right|_{in-plane} = 2\,R = \epsilon_I - \epsilon_{II} = 1484.92\,\mu \quad (35)$$

**Step VII. Show results on a differential element**

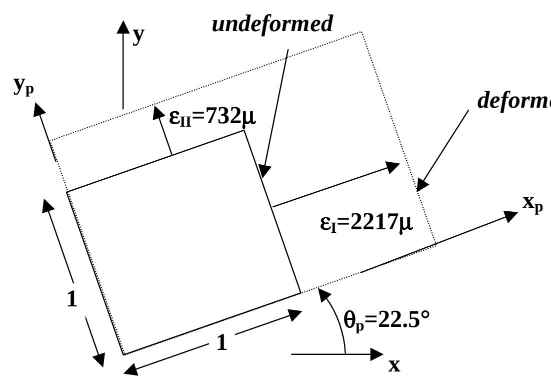


**Fig. 3  Principal strain axes shown with a deformed and undeformed element**