AOE 3024: Thin Walled Structures
Solutions to Homework # 7

Consider a cantilever beam as shown in the attached figure. At the tip of the beam, a bending moment $M = 1000$ N-m is applied at an angle θ with respect to the positive x-axis.

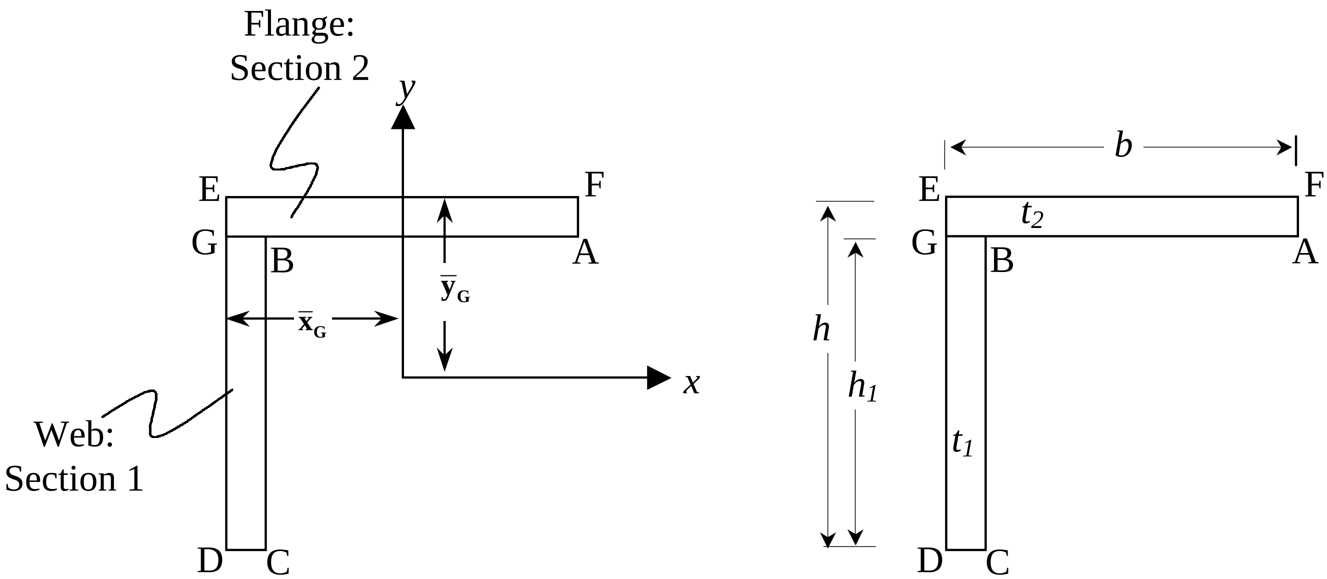

Fig. 1 Beam's Cross Section

Part A. For this beam, determine the maximum bending stress as the angle θ is varied between $0°$ to $180°$. Plot your results. Take $b = h = 100$ mm and $t_1 = t_2 = 5$ mm.

For this part no thin-wall assumption was used. The first step is to calculate the centroid. Let's place the the origin at point E and calculate $\overline{x}_G$ and $\overline{y}_G$.

Section	$\overline{x}_i$ mm	$\overline{y}_i$ mm	A_i mm^2
1	$t_1/2$ =2.5	$-(h_1/2 + t_2) = -52.5$	$(h - t_2)(t_1)$=475
2	$b/2$ =50	$-t_1/2 = -2.5$	$(b)(t_2)$=500

$$\sum A_i = ht_1 + bt_2 - t_1 t_2 = 975 \text{ mm}^2$$
$$\sum \overline{x}_i A_i = \frac{ht_1^2}{2} + \frac{b^2 t_2}{2} - \frac{t_1^2 t_2}{2} = 26187.5 \text{ mm}^3$$
$$\sum \overline{y}_i A_i = -\frac{ht_1^2}{2} - \frac{bt_1 t_2}{2} + \frac{t_1 t_2^2}{2} = -26187.5 \text{ mm}^3$$

$$\overline{x}_G = \frac{\sum \overline{x}_i A_i}{\sum A_i} = 26.859 \text{ mm} \qquad \overline{y}_G = \frac{\sum \overline{y}_i A_i}{\sum A_i} = -26.859 \text{ mm}$$

The minus sign indicates that the centroid is located below point E. We will relocate the axis as shown in Figure (1) and take discard the sign:

$$\overline{x}_G = 26.859 \text{ mm} \qquad \overline{y}_G = 26.859 \text{ mm} \tag{1}$$

Now we proceed to calculate the second moments of area. For simplicity take $h_1 = h - t_2$ (see figure (1)) Please see Mathematica file for details

Section	$I_{x_{ci}}$	$I_{y_{ci}}$	$I_{x_c y_{ci}}$
1	$(t_1 h_1^3)/12$	$(t_1^3 h_1)/12$	0
2	$(t_2 b^3)/12$	$(t_2^3 b)/12$	0

Second Moment of area $I_{xx} = I_{xx_1} + I_{xx_2}$ is

$$I_{xx_1} = I_{x_{c1}} + A_1 \left[- \left(\frac{h_1}{2} - (\overline{y}_G - t_2) \right) \right]^2 \tag{2a}$$

$$I_{xx_2} = I_{x_{c2}} + A_2 \left(\overline{y}_G - \frac{t_2}{2} \right)^2 \tag{2b}$$

$$I_{xx} = I_{xx_1} + I_{xx_2} = 967256 \text{ mm}^4 \tag{2c}$$

Second Moment of area $I_{yy} = I_{yy_1} + I_{yy_2}$ is

$$I_{yy_1} = I_{y_{c_1}} + A_1 \left[- \left(\overline{x}_G - \frac{t_1}{2} \right) \right]^2 \tag{3a}$$

$$I_{yy_2} = I_{y_{c_2}} + A_2 \left(\frac{b}{2} - \overline{x}_G \right)^2 \tag{3b}$$

$$I_{yy} = I_{yy_1} + I_{yy_2} = 967256 \text{ mm}^4 \tag{3c}$$

Second Moment of area $I_{xy} = I_{xy_1} + I_{xy_2}$ is

$$I_{xy_1} = I_{x_c y_{c_1}} + A_1 \left[- \left(\frac{h_1}{2} - (\overline{y}_G - t_2) \right) \right] \left[- \left(\overline{x}_G - \frac{t_1}{2} \right) \right] \tag{4a}$$

$$I_{xy_2} = I_{x_c y_{c_2}} + A_2 \left[\left(\overline{y}_G - \frac{t_2}{2} \right) \right] \left[\left(\frac{b}{2} - \overline{x}_G \right) \right] \tag{4b}$$

$$I_{xy} = I_{xy_1} + I_{xy_2} = 578526 \text{ mm}^4 \tag{4c}$$

Now we need to decompose the applied moment

$$M_x = M \cos\theta \qquad M_y = -M \sin\theta \tag{5}$$

and use equation 9.6 from Megson's text. (Note that the minus sign in M_y is to be consistent with the derivation of Eq. 3.6 of your text)

$$\begin{aligned} \sigma_{zz} &= \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\, x + \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\, y && (6) \\ &= (-962.779\, x \cos\theta + 1609.7\, y \cos\theta - 1609.7\, x \sin\theta + 962.779\, y \sin\theta) \text{ MPa} && (7) \end{aligned}$$

The above equation is evaluated at Points A, B, C, D, E, and F. See Mathematica file.

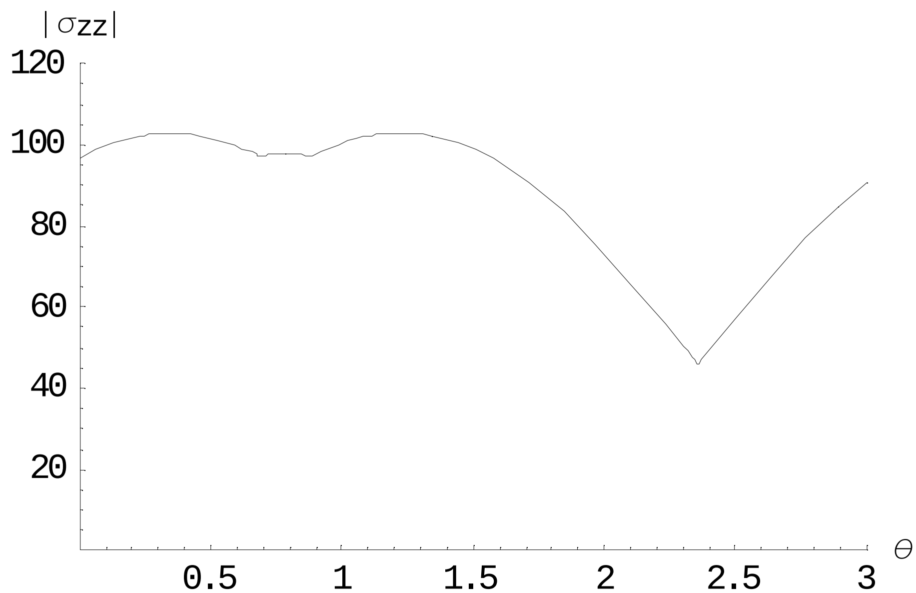

Fig. 2 Magnitude of the maximum bending stress acting in the cross section

Maximum bending stress occurs at location A at an angle 69.96° and has a value of 102.909 MPa in compression.

Maximum bending stress also occurs at location C at an angle 20.00° and has a value of 102.909 MPa in compression.

Part B. Use thin-wall assumption. For $\theta = 45°$, $b = 100$ mm, and $t_2 = 5$ mm, we want to study the effect of changing the ratio α between 0.70 and 1.30 while keeping the cross-section area of the beam constant, i.e., keep $\alpha\beta = 1$.

$$\alpha = \frac{h}{b} \quad \Rightarrow \quad h = b\,\alpha$$

$$\beta = \frac{t_1}{t_2} \quad \alpha\beta = \alpha\frac{t_1}{t_2} = 1 \quad \Rightarrow \quad t_1 = \frac{t_2}{\alpha}$$

In thin-wall assumption it is reasonable to ignore higher order thickness terms. Basically, substitute the h and t_1 for the above expressions and expand. Also, note that for thin-walled beam:

- Point A and point F are located at A'= F
- Point E and point B are located at B'= E
- Point C and point D are located at C'= D

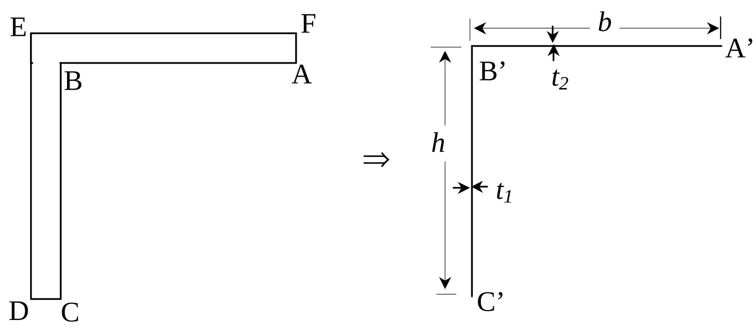

Fig. 3 Thin-walled assumption

The first step is to calculate the centroid. Note that the centroid will be a function of α. Let's place the the origin at point E and calculate $\overline{x}_G$ and $\overline{y}_G$. See Mathematica file.

Section	$\overline{x}_i$ mm	$\overline{y}_i$ mm	A_i mm^2
1	0	$-h/2$	$(h)(t_1)$
2	$b/2$	0	$(b)(t_2)$

$$\sum A_i = 2\,b\,t_2 = 1000 \text{ mm}^2$$

$$\sum \overline{x}_i\,A_i = \frac{b^2\,t_2}{2}$$

$$\sum \overline{y}_i\,A_i = -\left(\frac{b^2\,\alpha\,t_2}{2}\right)$$

$$
\begin{aligned}
\overline{x}_G &= \frac{\sum \overline{x}_i\, A_i}{\sum A_i} = \frac{b}{4} = 25.0 \text{ mm} \\
\overline{y}_G &= \frac{\sum \overline{y}_i\, A_i}{\sum A_i} = -\left(\frac{b\,\alpha}{4}\right) = -25.\,\alpha \text{ mm}
\end{aligned}
$$

The minus sign indicates that the centroid is located below point E. We will relocate the axis as shown in Figure (1) and discard the sign:

$$
\overline{x}_G = 25 \text{ mm} \qquad \overline{y}_G = 25\,\alpha \text{ mm} \tag{8}
$$

Now we proceed to calculate the second moments of area. Please see Mathematica file for details. Only the linearized expressions are given here

Second moments of area are

$$
I_{xx_1} = I_{x_{c1}} + A_1 \left[-\left(\frac{h}{2} - (\overline{y}_G)\right)\right]^2 \tag{9a}
$$

$$
I_{xx_2} = I_{x_{c2}} + A_2\,(\overline{y}_G)^2 \tag{9b}
$$

$$
I_{xx} = I_{xx_1} + I_{xx_2} \approx \frac{5\,b^3\,\alpha^2\,t_2}{24} = 1.04167 \times 10^6\,\alpha^2 \text{ mm}^4 \tag{9c}
$$

$$
I_{yy_1} = I_{y_{c1}} + A_1\,[-(\overline{x}_G)]^2 \tag{10a}
$$

$$
I_{yy_2} = I_{y_{c2}} + A_2 \left(\frac{b}{2} - \overline{x}_G\right)^2 \tag{10b}
$$

$$
I_{yy} = I_{yy_1} + I_{yy_2} \approx \frac{5\,b^3\,t_2}{24} = 1.04167 \times 10^6 \text{ mm}^4 \tag{10c}
$$

$$
I_{xy_1} = I_{x_c\,y_{c1}} + A_1 \left[-\left(\frac{h_1}{2} - (\overline{y}_G)\right)\right] [-(\overline{x}_G)] \tag{11a}
$$

$$
I_{xy_2} = I_{x_c\,y_{c2}} + A_2\,[(\overline{y}_G)] \left[\left(\frac{b}{2} - \overline{x}_G\right)\right] \tag{11b}
$$

$$
I_{xy} = I_{xy_1} + I_{xy_2} \approx \frac{b^3\,t_2\,\alpha}{8} = 625000.\,\alpha \text{ mm}^4 \tag{11c}
$$

Now we need to decompose the applied moment

$$
M_x = M\,\cos 45^\circ \qquad M_y = -M\,\sin 45^\circ \tag{12}
$$

and use equation 9.6 from your text

$$\sigma_{zz} = \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\, x + \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\, y \tag{13}$$

$$= x\left(\frac{-15\,M}{2\sqrt{2}\,b^3\,t_2} - \frac{9\,M}{2\sqrt{2}\,b^3\,\alpha\,t_2}\right) + y\left(\frac{15\,M}{2\sqrt{2}\,b^3\,\alpha^2\,t_2} + \frac{9\,M}{2\sqrt{2}\,b^3\,\alpha\,t_2}\right) \tag{14}$$

$$= (-1060.66\,x + \frac{1060.66}{\alpha^2}\,y - \frac{636.396}{\alpha}\,x + \frac{636.396}{\alpha}\,y)\ \text{MPa} \tag{15}$$

The above equation is evaluated at Points A', B', and C'. See Mathematica file.

Maximum bending stress (compression) occurs at location C' when $\alpha = 0.7$. This maximum stress has a value of 191.491 MPa.

Maximum bending stress (tension) occurs at location B' when $\alpha = 0.7$. This maximum stress has a value of 115.884 MPa.

Therefore, the maximum bending stress occurs at location C',

$$\sigma_{zz}\Big|_{\alpha=0.7} = 191.491\ \text{MPa, in compression} \tag{16}$$

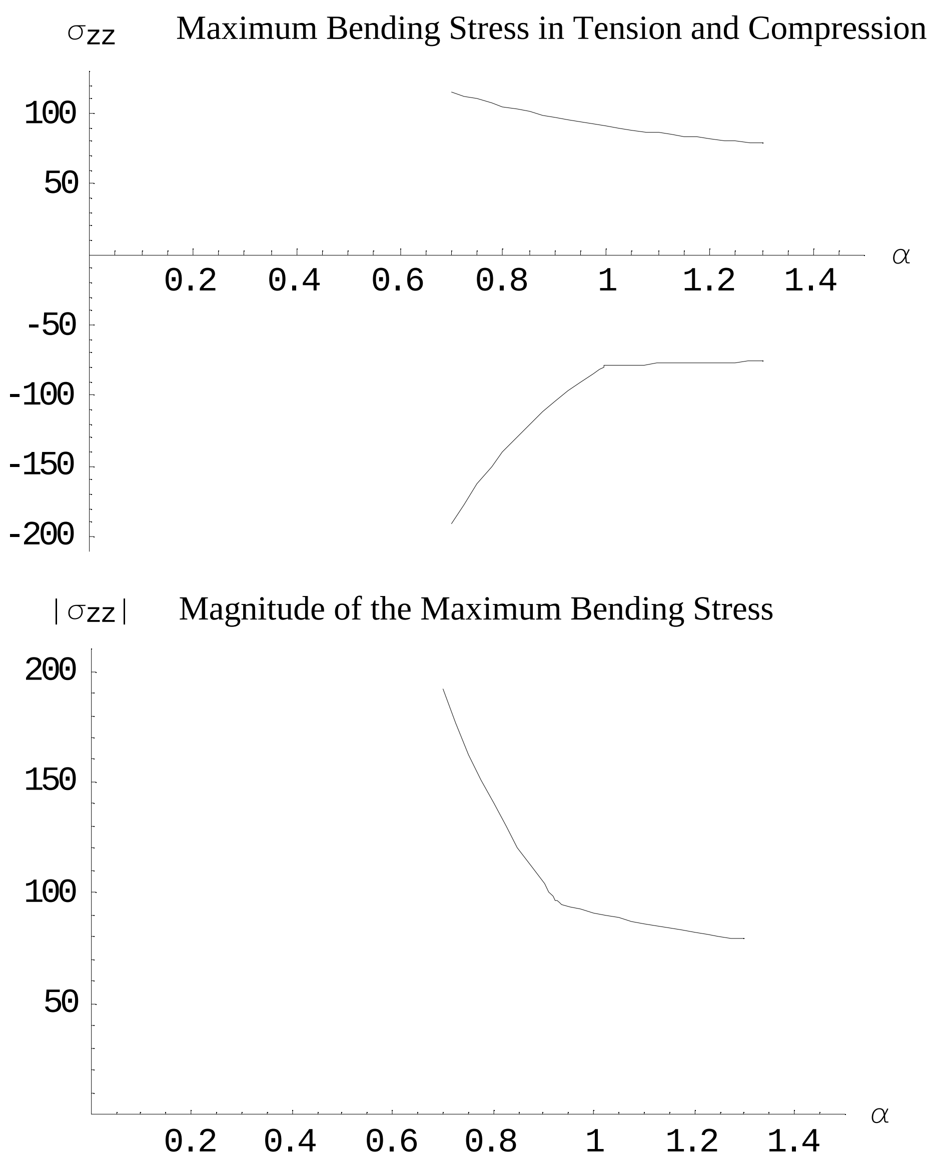

Fig. 4 Maximum bending stress acting in the cross section