AOE 3024: Thin Walled Structures
Solutions to Homework # 8

Consider a thin-walled cantilever beam as shown in the attached figure. At the tip of the beam, a bending moment $M = 3000$ N-m is applied at an angle θ with respect to the positive x-axis. Assume that the height of the web is h, the width of the flange is b, thickness of the web is t_1, and the thickness of the flange t_2. Take $b = h = 120$ mm and $t_1 = t_2 = 3$ mm.

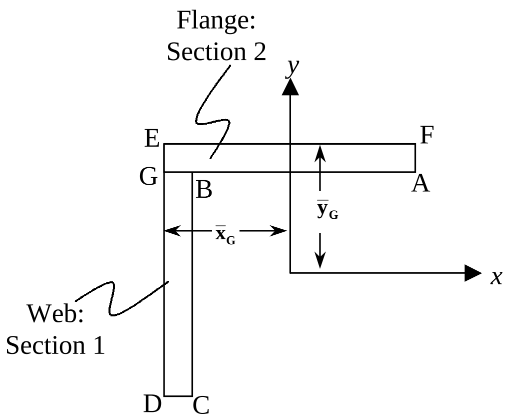

Fig. 1 Beam's Cross Section

For simplicity, let's define a new parameter β as the ratio of thicknesses,

$$\beta = \frac{t_2}{t_1} \tag{1}$$

This will help us to easily linearize the equations for this thin-walled beam. For our problem $\beta = 1$. In thin-wall assumption it is reasonable to ignore higher order thickness terms. Basically, substitute t_2 for $\beta\, t_1$, expand and ignore all quadratic and higher order terms in t_1. (We could have also substituted for t_1.) Also, note that for thin-walled beam: points A and F are located at A'= F; points E and point B are located at B'= E, and points C and D are located at C'= D.

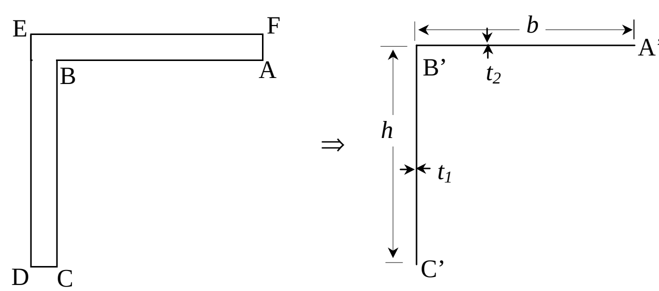

Fig. 2 Thin-walled assumption

Part A. For this beam, determine the resultant deflection vector at the tip and the mid-point as the angle θ is varied between 0° to 180°. Plot your results.

The first step is to calculate the centroid. Let's place the origin at point B' and calculate $\overline{x}_G$ and $\overline{y}_G$.

Section	$\overline{x}_i$	$\overline{y}_i$	A_i
1	0	$-h/2$	$(h)(t_1)$
2	$b/2$	0	$(b)(t_2)$

$$\sum A_i = (h + b\,\beta)\,t_1 \qquad \sum \overline{x}_i\,A_i = \frac{b^2\,\beta\,t_1}{2} \qquad \sum \overline{y}_i\,A_i = -\left(\frac{h^2\,t_1}{2}\right)$$

$$\begin{aligned}
\overline{x}_G &= \frac{\sum \overline{x}_i\,A_i}{\sum A_i} = \frac{b^2\,\beta}{2\,h + 2\,b\,\beta} = 30 \text{ mm} \\
\overline{y}_G &= \frac{\sum \overline{y}_i\,A_i}{\sum A_i} = -\left(\frac{h^2}{2\,h + 2\,b\,\beta}\right) = -30 \text{ mm}
\end{aligned}$$

(Compare to $\overline{x}_G = 31.1203$ mm and $\overline{y}_G = -31.1203$ mm without using thin-walled assumption)

The minus sign indicates that the centroid is located below point B'. We will relocate the axis as shown in Figure 1 and take discard the sign:

$$\overline{x}_G = 30 \text{ mm} \qquad \overline{y}_G = 30 \text{ mm} \tag{2}$$

Now we proceed to calculate the second moments of area. Please see Mathematica file for details. Only the linearized expressions are given here.

Section	$I_{x_{ci}}$	$I_{y_{ci}}$	$I_{x_c\,y_{ci}}$
1	$(t_1 h_1^3)/12$	$(t_1^3 h_1)/12 \approx 0$	0
2	$(t_2 b^3)/12$	$(t_2^3 b)/12 \approx 0$	0

Second Moment of area I_{xx} using thin-walled assumption is

$$I_{xx_1} = I_{x_{c1}} + A_1\left[-\left(\frac{h}{2} - (\overline{y}_G)\right)\right]^2 \tag{3a}$$

$$I_{xx_2} = I_{x_{c2}} + A_2\,(\overline{y}_G)^2 \tag{3b}$$

$$I_{xx} = I_{xx_1} + I_{xx_2} \approx \frac{h^4\,t_1}{12\,h + 12\,b\,\beta} + \frac{4\,b\,h^3\,\beta\,t_1}{12\,h + 12\,b\,\beta} = 1.08 \times 10^6 \text{ mm}^4 \tag{3c}$$

(Compare to $I_{xx} = 1.04 \times 10^6$ mm^4 without using thin-walled assumption)

Second Moment of area I_{yy} using thin-walled assumption is

$$I_{yy_1} = I_{y_{c1}} + A_1 \left[-\left(\overline{x}_G\right)\right]^2 \tag{4a}$$

$$I_{yy_2} = I_{y_{c2}} + A_2 \left(\frac{b}{2} - \overline{x}_G\right)^2 \tag{4b}$$

$$I_{yy} = I_{yy_1} + I_{yy_2} \approx \frac{4\, b^3\, h\, \beta\, t_1}{12\, h + 12\, b\, \beta} + \frac{b^4\, \beta^2\, t_1}{12\, h + 12\, b\, \beta} = 1.08 \times 10^6 \text{ mm}^4 \tag{4c}$$

(Compare to $I_{yy} = 1.04 \times 10^6$ mm^4 without using thin-walled assumption)

Second Moment of area I_{xy} using thin-walled assumption is

$$I_{xy_1} = I_{x_c\, y_{c1}} + A_1 \left[-\left(\frac{h}{2} - (\overline{y}_G)\right)\right] \left[-\left(\overline{x}_G\right)\right] \tag{5a}$$

$$I_{xy_2} = I_{x_c\, y_{c2}} + A_2 \left[(\overline{y}_G)\right] \left[\left(\frac{b}{2} - \overline{x}_G\right)\right] \tag{5b}$$

$$I_{xy} = I_{xy_1} + I_{xy_2} \approx \frac{b^2\, h^2\, \beta\, t_1}{4\, h + 4\, b\, \beta} = 0.648 \times 10^6 \text{ mm}^4 \tag{5c}$$

(Compare to $I_{xy} = 0.623803 \times 10^6$ mm^4 without using thin-walled assumption)

Now we need to decompose the applied moment

$$M_x = M \cos\theta \qquad M_y = -M \sin\theta \tag{6}$$

where M is the applied moment at an angle θ from the x-axis. Note that the bending moment is uniform everywhere. (Note that the minus sign in M_y is to be consistent with the derivation of Eq. 9.17 of your text) Now using equation 9.17 from Megson's text

$$u''(z) = \frac{M_x I_{xy} - M_y I_{xx}}{E\left(I_{xx} I_{yy} - I_{xy}^2\right)} \tag{7}$$

$$v''(z) = \frac{M_y I_{xy} - M_x I_{yy}}{E\left(I_{xx} I_{yy} - I_{xy}^2\right)} \tag{8}$$

(Note that for the present problem the right hand side of the above equations is constant and only a function of θ)

Integrating Eq. (7) respect to z:

$$u'(z) = \int u''(z)\, dz + A \tag{9a}$$

$$u'(z) = \int \frac{M_x I_{xy} - M_y I_{xx}}{E\left(I_{xx} I_{yy} - I_{xy}^2\right)}\, dz + A \tag{9b}$$

$$u'(z) = \frac{M_x I_{xy} - M_y I_{xx}}{E\left(I_{xx} I_{yy} - I_{xy}^2\right)}\, z + A \tag{9c}$$

(9d)

Integrating Eq. (9) respect to z:

$$u(z) = \int u'(z)\ dz + B \tag{10a}$$

$$u(z) = \int \frac{M_x I_{xy} - M_y I_{xx}}{E\left(I_{xx} I_{yy} - I_{xy}^2\right)}\ z\ dz + A\,z + B \tag{10b}$$

$$u(z) = \frac{M_x I_{xy} - M_y I_{xx}}{E\left(I_{xx} I_{yy} - I_{xy}^2\right)}\ \frac{z^2}{2} + A\,z + B \tag{10c}$$

The integration constants A and B are found applying the boundary conditions:

$$u'(z)\Big|_{z=0} = 0 \quad \Rightarrow \quad A = 0 \tag{11}$$

$$u(z)\Big|_{z=0} = 0 \quad \Rightarrow \quad B = 0 \tag{12}$$

Therefore, the horizontal component of deflection using thin-walled assumption is

$$u(z) = \left\{\frac{M_x I_{xy} - M_y I_{xx}}{E\left(I_{xx} I_{yy} - I_{xy}^2\right)}\right\} \frac{z^2}{2} \tag{13a}$$

$$= \frac{3\,M\ \left(3\,b^2\,\beta\ \cos\theta + h\ (h + 4\,b\,\beta)\ \sin\theta\right)}{2\,E\,b^3\,h\,\beta\ (h + b\,\beta)\ t_1}\ z^2 \tag{13b}$$

$$= \left(6.51042 \times 10^{-6}\ \cos\theta + 10.8507 \times 10^{-6}\ \sin\theta\right)\ z^2 \text{ mm, for } z \text{ in [mm]} \tag{13c}$$

(Compare to $u(z) = (6.74673 \times 10^{-6}\ \cos\theta + 11.2532 \times 10^{-6}\ \sin\theta)\ z^2$ without using thin-walled assumption.) Also, recall that $\beta = 1$.

Similarly, the vertical component of deflection is

$$v(z) = \left\{\frac{M_y I_{xy} - M_x I_{yy}}{E\left(I_{xx} I_{yy} - I_{xy}^2\right)}\right\} \frac{z^2}{2} \tag{14a}$$

$$= -\frac{3\,M\ \left(b\ (4\,h + b\,\beta)\ \cos\theta + 3\,h^2\ \sin\theta\right)}{2\,b\,h^3\ (h + b\,\beta)\ t_1}\ z^2 \tag{14b}$$

$$= \left(-10.8507 \times 10^{-6}\ \cos\theta - 6.51042 \times 10^{-6}\ \sin\theta\right)\ z^2 \text{ mm, for } z \text{ in [mm]} \tag{14c}$$

(Compare to $v(z) = (-11.2532 \times 10^{-6}\ \cos\theta - 6.74673 \times 10^{-6}\ \sin\theta)\ z^2$ without using thin-walled assumption)

The magnitude and angle of the deflection vector is

$$\delta_t(z) = \sqrt{[u(z)]^2 + [v(z)]^2} = z^2 \times 10^{-5}\ \sqrt{1.60123 + 1.41285\ \sin{(2\theta)}} \text{ mm, for } z \text{ in [mm]} \tag{15}$$

$$\alpha_t = \tan^{-1}\left\{\frac{v(z)}{u(z)}\right\} = \frac{180}{\pi}\ \tan^{-1}\left\{-0.6 - \frac{1}{0.9375 + 1.5625\ \tan\theta}\right\} \text{ deg}$$

(Note that the angle does not depend on the location in z)

At the tip, $z = L$:

$$u_{tip} = 14.6484 \cos\theta + 24.4141 \sin\theta \qquad \text{mm} \tag{16}$$

$$v_{tip} = -24.4141 \cos\theta - 14.6484 \sin\theta \qquad \text{mm} \tag{17}$$

$$\delta_{tip} = \sqrt{810.623 + 715.256 \sin(2\,\theta)} \text{ mm} \tag{18}$$

$$\alpha_{tip} = \frac{180}{\pi} \tan^{-1}\left\{-0.6 - \frac{1}{0.9375 + 1.5625 \tan\theta}\right\} \text{ deg} \tag{19}$$

At the middle, $z = L/2$:

$$u_{mid} = 3.66211 \cos\theta + 6.10352 \sin\theta \qquad \text{mm} \tag{20}$$

$$v_{mid} = -6.10352 \cos\theta - 3.66211 \sin\theta \qquad \text{mm} \tag{21}$$

$$\delta_{mid} = \sqrt{50.6639 + 44.7035 \sin(2\,\theta)} \text{ mm} \tag{22}$$

$$\alpha_{mid} = \frac{180}{\pi} \tan^{-1}\left\{-0.6 - \frac{1}{0.9375 + 1.5625 \tan\theta}\right\} \text{ deg} \tag{23}$$

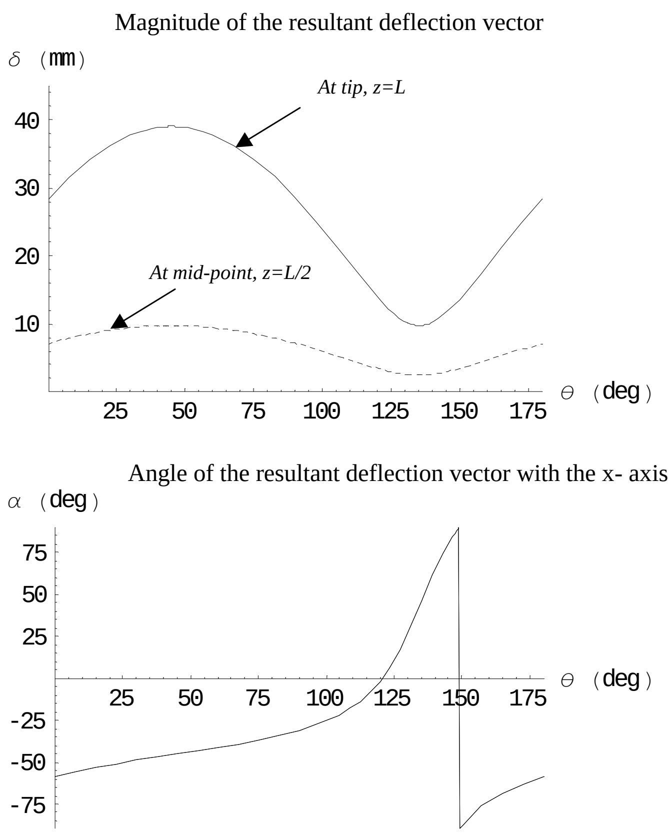

Fig. 3 Resultant deflection vector

Part B. For both the tip and the mid-point of the beam, plot the angle that the deflection vector makes with the neutral axis as the angle θ is varied.

The neutral axis is where the bending stress is zero. Therefore, we use equation 9.6 from your text

$$\begin{aligned}
\sigma_{zz} &= \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\, x + \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\, y && \text{(24a)} \\
&= \frac{3\,M\,\left(-3\,b^2\,h^2\,\beta\,\cos\theta - h^4\,\sin\theta - 4\,b\,h^3\,\beta\,\sin\theta\right)}{b^3\,h^3\,\beta\,\left(h + b\,\beta\right)\,t_1}\, x && \text{(24b)} \\
&\quad + \frac{3\,M\,\left(4\,b^3\,h\,\beta\,\cos\theta + b^4\,\beta^2\,\cos\theta + 3\,b^2\,h^2\,\beta\,\sin\theta\right)}{b^3\,h^3\,\beta\,\left(h + b\,\beta\right)\,t_1}\, y \\
&= \left(-2604.17\,\cos\theta - 4340.28\,\sin\theta\right) x + \left(4340.28\,\cos\theta + 2604.17\,\sin\theta\right) y \text{ MPa} && \text{(24c)}
\end{aligned}$$

Setting the stress to zero, we get an equation for the neutral axis:

$$\begin{aligned}
0 &= \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\, x + \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\, y \\
y &= -\frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}}\, x && \text{(25)} \\
y &= \frac{2604.17\,\cos\theta + 4340.28\,\sin\theta}{4340.28\,\cos\theta + 2604.17\,\sin\theta}\, x
\end{aligned}$$

Now, the slope of the above equation is the neutral axis angle

$$\psi = \frac{180}{\pi} \tan^{-1} \left\{ \frac{h^2\,\left(3\,b^2\,\beta\,\cos\theta + h\,\left(h + 4\,b\,\beta\right)\,\sin\theta\right)}{b^2\,\beta\,\left(b\,\left(4\,h + b\,\beta\right)\,\cos\theta + 3\,h^2\,\sin\theta\right)} \right\} \text{ deg} \tag{26}$$

The angle that the deflection vector makes with the neutral axis is given by $\alpha - \psi$. The variation of $\alpha - \psi$ as a function of θ is shown in Figure 4. Note that the difference is $\pi/2$ for all values of θ. The deflection vector is thus normal to the axis.

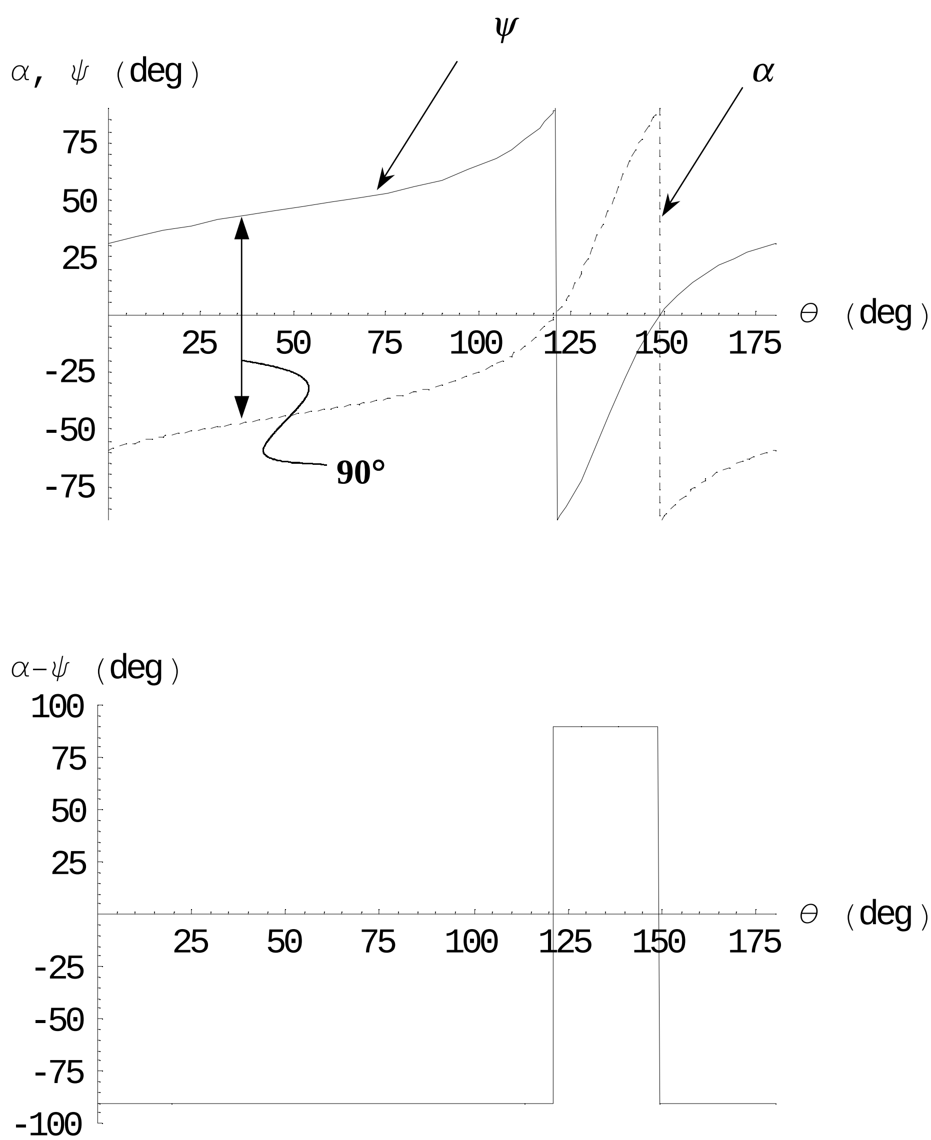

Fig. 4 Resultant deflection angle about the neutral axis as angle θ is varied. Note that the difference is always 90°.