AOE 3024: Thin Walled Structures
Solutions to Homework # 9

A beam has the singly symmetrical, thin-walled cross-section shown. Each wall of the section is flat and has the same length a and thickness t. Calculate the distance of the shear center from the point 3.

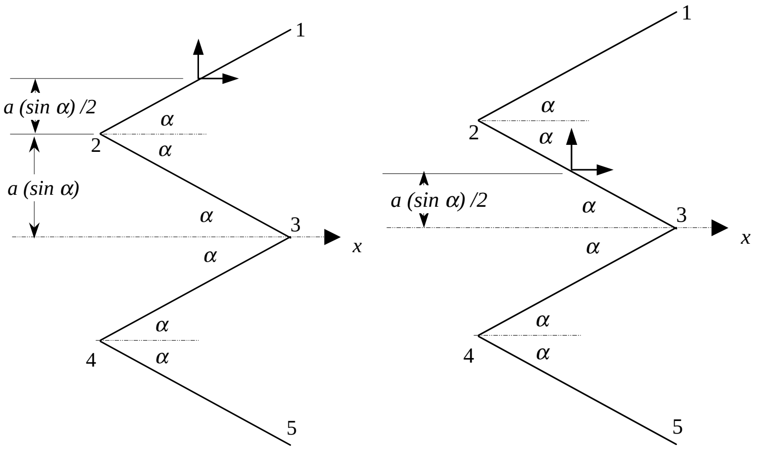
First, calculate the second moment of areas

Always consider calculating the product second moment of area first because whenever there is an axis of symmetry, $I_{xy} = 0$. Thus in this problem the product second moment of area is zero.

For thin-walled sections, the second moment of area about the horizontal axis through the centroid of an inclined section is given in page 288 of the course textbook,

$$I_{xx} = \frac{a^3\, t\, \sin^2\alpha}{12}$$

Now we calculate I_{xx} for each section and add them up.



a) For section 1-2 **b)** For section 2-3

Fig. 1

$$\begin{aligned} I_{xx_{12}} &= I_{x_{c12}} + A_{12}\, dy_{12}^2 \\ &= \frac{a^3\, t\, \sin^2\alpha}{12} + (a\, t)\left[a\, \sin\alpha + \frac{a}{2}\, \sin\alpha\right]^2 \\ &= \frac{7}{3}\, a^3\, t\, \sin^2\alpha \end{aligned} \tag{1a}$$

$$
\begin{aligned}
I_{xx_{23}} &= I_{x_{c23}} + A_{23}\, dy_{23}^2 \\
&= \frac{a^3\, t\, \sin^2(-\alpha)}{12} + (a\, t)\left[\frac{a}{2}\, \sin(-\alpha)\right]^2 \\
&= \frac{1}{3}\, a^3\, t\, \sin^2\alpha
\end{aligned}
\tag{2a}
$$

Because of symmetry:

$$
\begin{aligned}
I_{xx_{54}} &= I_{x_{c54}} + A_{54}\, dy_{54}^2 \\
&= I_{xx_{12}} = \frac{7}{3}\, a^3\, t\, \sin^2\alpha
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
I_{xx_{43}} &= I_{x_{c43}} + A_{43}\, dy_{43}^2 \\
&= I_{xx_{23}} = \frac{1}{3}\, a^3\, t\, \sin^2\alpha
\end{aligned}
\tag{4}
$$

Thus the second moment of area about the x-axis is

$$I_{xx} = I_{xx_{12}} + I_{xx_{23}} + I_{xx_{43}} + I_{xx_{54}} = \frac{16}{3}\, a^3\, t\, \sin^2\alpha \tag{5}$$

For thin-walled sections, the second moment of area about the vertical axis through the centroid of an inclined section is given in page 288 of the course textbook,

$$I_{yy} = \frac{a^3\, t\, \cos^2\alpha}{12}$$

However, in this problem it is not necessary to calculate I_{yy}.

One can often reduce the amount of computation by giving some thought to the problem.

Second, calculate the shear flow distribution

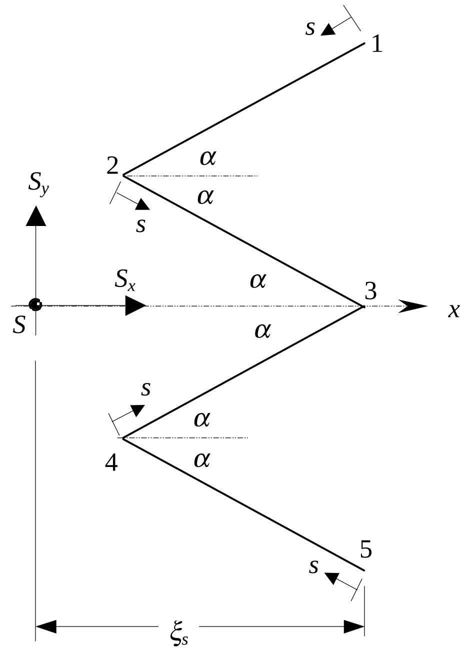


Fig. 2 Shear flow convention used in this problem

Note that the x-axis is an axis of symmetry. Thus the shear center S lies on this axis and $S_x = 0$. Therefore, an arbitrary shear flow force S_y is applied through S and the internal shear flow distribution determined.

Since S_y is applied through the shear center then no torsion exists and the shear flow distribution is

$$q_i(s) = q_{o_i} - \left(\frac{S_x\, I_{xx} - S_y\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s t_i(s_i)\, x_i(s_i)\, ds_i - \left(\frac{S_y\, I_{yy} - S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s t_i(s_i)\, y_i(s_i)\, ds_i \qquad (6)$$

where q_{o_i} is found by evaluating $q_i(s = 0)$.

If the origin for s is taken at the open edge of the cross-section,

$$q_i(s)\Big|_{(s=0)} = 0 \qquad \Rightarrow \qquad q_{o_i} = 0$$

When origin for s is taken at the open edge of the cross-section and the thickness is constant along each branch ($t_i(s_i) = t_i$) the equation is given by Eq. 9.34 (of the course textbook)

$$q_i(s) = -\left(\frac{S_x\, I_{xx} - S_y\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s t_i\, x_i(s_i)\, ds_i - \left(\frac{S_y\, I_{yy} - S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s t_i\, y_i(s_i)\, ds_i$$

However, it is suggested that Eq. (6) be always used.

For a section having either Cx or Cy as an axis of symmetry, $I_{xy} = 0$, and the above equation reduces to,

$$q_i(s) = q_{o_i} - \left(\frac{S_x}{I_{yy}}\right) \int_0^s t_i\, x_i(s_i)\, ds_i - \left(\frac{S_y}{I_{xx}}\right) \int_0^s t_i\, y_i(s_i)\, ds_i \tag{7}$$

When $S_x = 0$, $I_{xy} = 0$ and $t_i(s_i) = t_i$, the equation reduces to

$$q_i(s) = q_{o_i} - \frac{S_y}{I_{xx}} \int_0^s t_i\, y_i(s_i)\, ds_i \tag{8}$$

In order to calculate the shear flow in each flange, we need to first find the $y_i(s_i)$ for each part.Following the assumed flow convention in Fig. 2 and from the geometry of the cross-section we get:

For flange 54,

$$s_{54} = 0 \qquad \Rightarrow \qquad y_{54} = -2\,a\,\sin\alpha \tag{9}$$

$$s_{54} = a \qquad \Rightarrow \qquad y_{54} = -a\,\sin\alpha \tag{10}$$

For flange 43,

$$s_{43} = 0 \qquad \Rightarrow \qquad y_{43} = -a\,\sin\alpha \tag{11}$$

$$s_{43} = a \qquad \Rightarrow \qquad y_{43} = 0 \tag{12}$$

For flange 32,

$$s_{23} = 0 \qquad \Rightarrow \qquad y_{23} = a\,\sin\alpha \tag{13}$$

$$s_{23} = a \qquad \Rightarrow \qquad y_{23} = 0 \tag{14}$$

For flange 21,

$$s_{12} = 0 \qquad \Rightarrow \qquad y_{12} = 2\,a\,\sin\alpha \tag{15}$$

$$s_{12} = a \qquad \Rightarrow \qquad y_{12} = a\,\sin\alpha \tag{16}$$

Since these are straight lines, the equation of the line to can be used to calculate $y_i(s_i)$,

$$y_i(s_i) = \frac{y_i\Big|_{(s_i=a)} - y_i\Big|_{(s_i=0)}}{a-0}\, s_i + b \qquad \text{where} \qquad b = y_i\Big|_{(s_i=0)} \tag{17}$$

For flange 54, $(t_{54} = t)$

$$\begin{aligned}
y_{54}(s_{54}) &= (s_{54} - 2a)\,\sin\alpha \\
q_{54}(s) &= q_{o_{54}} - \frac{S_y}{I_{xx}} \int_0^s t_{54}\, y_{54}(s_{54})\, ds_{54} = q_{o_{54}} - \frac{S_y}{I_{xx}} \int_0^s t\,(s_{54} - 2a)\,\sin\alpha\, ds_{54} \\
&= q_{o_{54}} - \frac{S_y\, t\, \sin\alpha}{I_{xx}} \left(\frac{s^2}{2} - 2\,a\,s \right) \\
q_{o_{54}} &= q_{54}(0) = 0 \qquad \text{free edge} \\
q_{54}(s) &= -\frac{S_y\, t\, \sin\alpha}{I_{xx}} \left(\frac{s^2}{2} - 2\,a\,s \right) \qquad (18)
\end{aligned}$$

For flange 43, $(t_{43} = t)$

$$\begin{aligned}
y_{43}(s_{43}) &= (s_{43} - a)\,\sin\alpha \\
q_{43}(s) &= q_{o_{43}} - \frac{S_y}{I_{xx}} \int_0^s t_{43}\, y_{43}(s_{43})\, ds_{43} = q_{o_{43}} - \frac{S_y}{I_{xx}} \int_0^s t\,(s_{43} - a)\,\sin\alpha\, ds_{43} \\
&= q_{o_{43}} - \frac{S_y\, t\, \sin\alpha}{I_{xx}} \left(\frac{s^2}{2} - a\,s \right) \\
q_{o_{43}} &= q_{43}(0) = q_{54}(a) = \frac{S_y\, t\, \sin\alpha}{I_{xx}} \left(\frac{3a^2}{2} \right) \\
q_{43}(s) &= -\frac{S_y\, t\, \sin\alpha}{I_{xx}} \left(\frac{s^2}{2} - a\,s - \frac{3a^2}{2} \right) \qquad (19)
\end{aligned}$$

For flange 12, $(t_{12} = t)$

$$\begin{aligned}
y_{12}(s_{12}) &= -(s_{12} - 2a)\,\sin\alpha \\
q_{12}(s) &= q_{o_{12}} - \frac{S_y}{I_{xx}} \int_0^s t_{12}\, y_{12}(s_{12})\, ds_{12} = q_{o_{12}} + \frac{S_y}{I_{xx}} \int_0^s t\,(s_{12} - 2a)\,\sin\alpha\, ds_{12} \\
&= q_{o_{12}} + \frac{S_y\, t\, \sin\alpha}{I_{xx}} \left(\frac{s^2}{2} - 2\,a\,s \right) \\
q_{o_{12}} &= q_{12}(0) = 0 \qquad \text{free edge} \\
q_{12}(s) &= \frac{S_y\, t\, \sin\alpha}{I_{xx}} \left(\frac{s^2}{2} - 2\,a\,s \right) \qquad (20)
\end{aligned}$$

For flange 23, ($t_{23} = t$)

$$\begin{aligned}
y_{23}(s_{23}) &= -(s_{23} - a)\,\sin\alpha \\
q_{23}(s) &= q_{o_{23}} - \frac{S_y}{I_{xx}} \int_0^s t_{23}\, y_{23}(s_{23})\, ds_{23} = q_{o_{23}} + \frac{S_y}{I_{xx}} \int_0^s t\,(s_{23} - a)\,\sin\alpha\, ds_{43} \\
&= q_{o_{23}} + \frac{S_y\, t\,\sin\alpha}{I_{xx}} \left(\frac{s^2}{2} - a\,s\right) \\
q_{o_{23}} &= q_{23}(0) = q_{12}(a) = -\frac{S_y\, t\,\sin\alpha}{I_{xx}} \left(\frac{3a^2}{2}\right) \\
q_{23}(s) &= \frac{S_y\, t\,\sin\alpha}{I_{xx}} \left(\frac{s^2}{2} - a\,s - \frac{3a^2}{2}\right) \qquad (21)
\end{aligned}$$

Check: at point 3 the static equilibrium must be satisfied,

$$\begin{aligned}
q_{23}(a) + q_{43}(a) &= \text{must be zero} \\
&= \frac{S_y\, t\,\sin\alpha}{I_{xx}} \left(\frac{a^2}{2} - a\,a - \frac{3a^2}{2}\right) + -\frac{S_y\, t\,\sin\alpha}{I_{xx}} \left(\frac{a^2}{2} - a\,a - \frac{3a^2}{2}\right) \\
&= 0 \text{ ...GOOD!}
\end{aligned}$$

Third, calculate the shear center

Torque equivalence about a point, say O_s, determines the location ξ_s of the shear center. For a positive torque counterclockwise (see Fig. 3),

$$-S_y\,\xi_s = \sum_i \int_0^a \vec{r}_i(s) \times \vec{q}_i(s)\, ds \qquad (22)$$

where i represents the i^{th} branch, and $r_i(s)$ the distance perpendicular from the point O_s to the contour s_i.

Suppose we take torque equivalence about point 3 ($O_s = 3$), then the lines of action of branches 43 and 23 pass through point 3 and thus there was no need to calculate the shear flow through them.

Equation (22) becomes

$$-S_y\,\xi_s = -\int_0^a r_{54}(s)\, q_{54}(s)\, ds + \int_0^a r_{12}(s)\, q_{12}(s)\, ds \qquad (23)$$

where $r_i(s)$ is found as shown in Fig. 3.

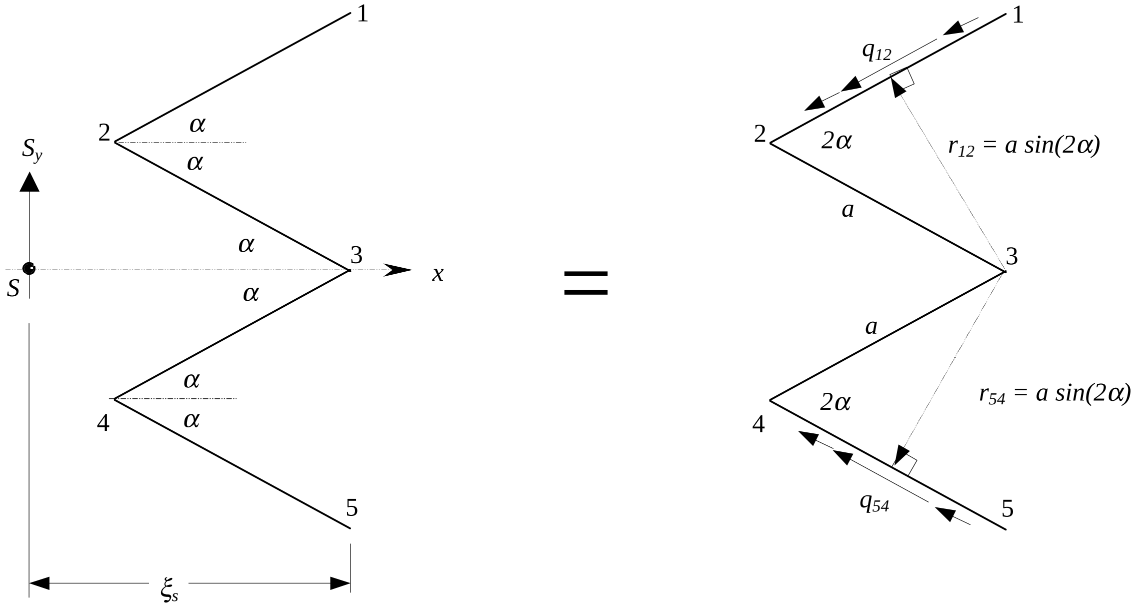


Fig. 3 Statically equivalent

Therefore,

$$\begin{aligned} -S_y\,\xi_s &= -\int_0^a q_{54}(s)\,a\,\sin 2\alpha\,ds + \int_0^a q_{12}(s)\,a\,\sin 2\alpha\,ds \\ &= \frac{2\,a\,\sin 2\alpha\,S_y\,t\,\sin\alpha}{I_{xx}}\int_0^a \left(\frac{s^2}{2} - 2\,a\,s\right)\,ds \\ &= -\frac{2\,a\,\sin 2\alpha\,S_y\,t\,\sin\alpha}{I_{xx}}\left(\frac{5}{6}\,a^3\right) \end{aligned} \tag{24}$$

Substituting I_{xx} from Eq. (5) and multiplying by -1

$$S_y\,\xi_s = \frac{2\,a\,\sin 2\alpha\,S_y\,t\,\sin\alpha}{\left(\frac{16}{3}\,a^3\,t\,\sin^2\alpha\right)}\left(\frac{5}{6}\,a^3\right)$$

Using trigonometric identity: $\sin(2\alpha) = 2\,\sin\alpha\,\cos\alpha$ and simplifying

$$\boxed{\xi_s = \frac{5}{8}\,a\,\cos\alpha}$$

Ans.

We could have taken torque equivalence about any point. Suppose we take torque equivalence about point 1 ($O_s = 1$), then the lines of action of branch 12 pass through point 1 and thus creates no moment.

Equation (22) becomes

$$-S_y\,\xi_s \;=\; -\int_0^a r_{54}(s)\,q_{54}(s)\,ds + \int_0^a r_{23}(s)\,q_{23}(s)\,ds + \int_0^a r_{43}(s)\,q_{43}(s)\,ds \tag{25}$$

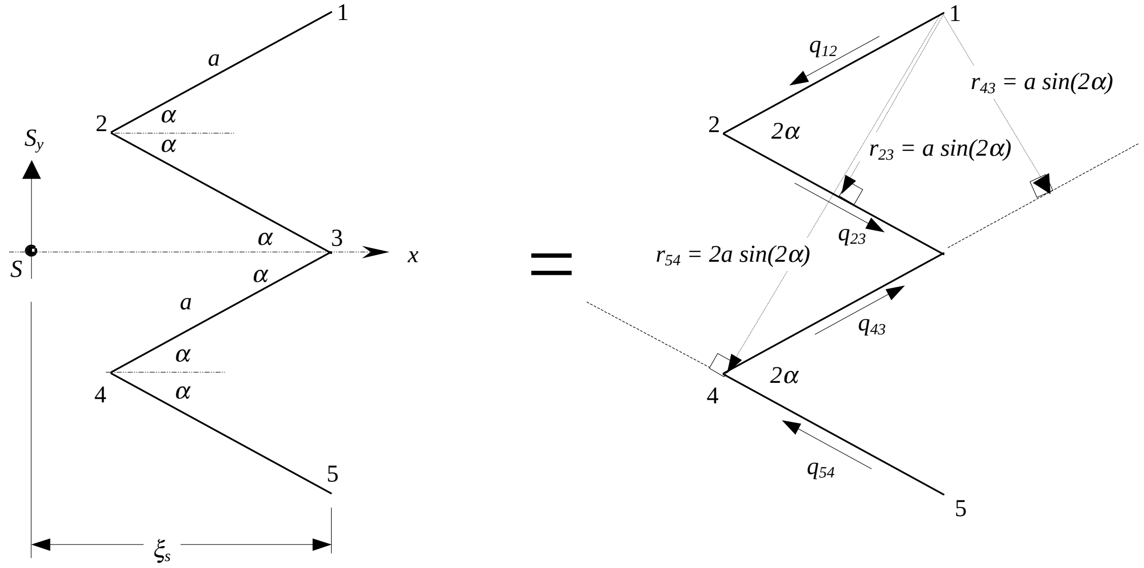


Fig. 4 Statically equivalent. (Note that the shear flow is distributed along the branches. However, for simplicity only one arrow is shown.)

where $r_i(s)$ is found as shown in Fig. 4.

Therefore,

$$\begin{aligned} -S_y\,\xi_s &= -\int_0^a q_{54}(s)\,2\,a\,\sin 2\alpha\,ds + \int_0^a q_{23}(s)\,a\,\sin 2\alpha\,ds + \int_0^a q_{43}(s)\,a\,\sin 2\alpha\,ds \\ &= -\frac{2\,a\,\sin 2\alpha\,S_y\,t\,\sin\alpha}{I_{xx}}\left(\frac{5}{6}\,a^3\right) \end{aligned} \tag{26}$$

Substituting I_{xx} from Eq. (5) and multiplying by -1

$$S_y\,\xi_s = \frac{2\,a\,\sin 2\alpha\,S_y\,t\,\sin\alpha}{\left(\frac{16}{3}\,a^3\,t\,\sin^2\alpha\right)}\left(\frac{5}{6}\,a^3\right)$$

Same as before!