6.1 Generic Approximations

The most commonly used approximations to objective functions and constraints are based on the value of the function and its derivatives at one or several points. Most of these approximations are applicable to any function, regardless of whether it describes structural response or not. For this reason we refer to such approximations as generic. Approximations that are specific to the form of analysis that is used to generate the function are dealt with in the next section. Generic approximations can be divided into local approximations, that are sufficiently accurate only in a limited region of the design space, and global approximations that attempt to approximate the function in the entire design space. Midrange approximations offer a compromise between the two.

6.1.1 *Local Approximations*

The simplest local approximation is the linear approximation based on the Taylor series. Given a function $g(\mathbf{x})$, the linear approximation $g_L(\mathbf{x})$ is

$$g_L(\mathbf{x}) = g(\mathbf{x}_0) + \sum_{i=1}^{n} (x_i - x_{0i}) \left(\frac{\partial g}{\partial x_i} \right)_{\mathbf{x}_0} . \tag{6.1.1}$$

For many applications the linear approximation is inaccurate even for design points $\mathbf{x}$ that are close to $\mathbf{x}_0$. Accuracy can be increased by retaining additional terms in the Taylor series expansion. This, however, requires the costly calculation of higher-order derivatives. A more attractive alternative is to find intervening variables that would make the approximated function behave more linearly. That is, define

$$y_i = y_i(\mathbf{x}) \qquad i = 1, \ldots, m , \tag{6.1.2}$$

where y_i are m functions of the design variables called intervening variables. The linear approximation, g_I, in terms of the intervening variables is

$$g_I(\mathbf{y}) = g(\mathbf{y}_0) + \sum_{i=1}^{m}(y_i - y_{0i})\left(\frac{\partial g}{\partial y_i}\right)_{\mathbf{y}_0} , \tag{6.1.3}$$

where $y_{0i} = y_i(\mathbf{x}_0)$, and the derivatives of g with respect to the y_i's can be calculated from the derivatives with respect to the x_i's.

Example 6.1.1

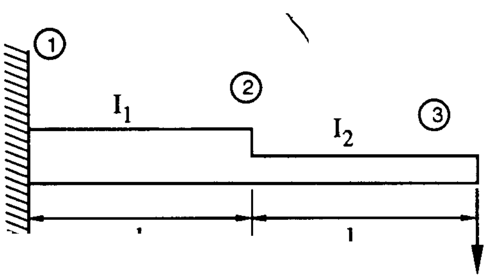

Figure 6.1.1 Beam example.

The beam shown in Fig. (6.1.1) has a rectangular cross section of width b_i and height h_i, $i = 1, 2$. The tip displacement is constrained not to exceed w_{all}; with elementary beam theory this constraint can be written as

$$g = w_{\text{all}} - \left(\frac{23}{6}\right)\frac{pl^3}{EI_1} - \left(\frac{5}{6}\right)\frac{pl^3}{EI_2} .$$

If the design variables are the width and height of each section, we can express g in terms of these design variables as

$$g = w_{\text{all}} - \frac{46pl^3}{Eb_1h_1^3} - \frac{10pl^3}{Eb_2h_2^3} .$$

This expression is a highly non-linear function of the design variables, but it can be linearized by using the intervening variables

$$y_1 = \frac{1}{I_1} = \frac{12}{b_1h_1^3} , \qquad \text{and} \qquad y_2 = \frac{1}{I_2} = \frac{12}{b_2h_2^3} .$$

The constraint function can then be written as a linear function

$$g = w_{\text{all}} - \left(\frac{23}{6}\right)\frac{pl^3}{E}y_1 - \left(\frac{5}{6}\right)\frac{pl^3}{E}y_2 .$$

• • •

The cases where intervening variables can exactly linearize the constraint are rather rare. Example (6.1.1) is typical of statically determinate structures where such linearization is often possible. However, as shown by Mills-Curran et al. [2], even in the case of statically indeterminate beam and frame structures, the reciprocals of moments of inertia are good intervening variables for displacement constraints.

In many applications the intervening variables are functions of a single design variable, that is

$$y_i = y_i(x_i) \qquad i = 1, \ldots, n \; . \tag{6.1.4}$$

In this case it is often convenient to write g_I, Eq. (6.1.3), in terms of the original variables

$$g_I(\mathbf{x}) = g(\mathbf{x}_0) + \sum_{i=1}^{n} \Big(y_i(x_i) - y_i(x_{0i})\Big) \left(\frac{\partial g}{\partial x_i} / \frac{dy_i}{dx_i}\right)_{\mathbf{x}_0} \; . \tag{6.1.5}$$

Note that while g_I is a linear function of $\mathbf{y}$ it is, in general, a nonlinear function of $\mathbf{x}$.

One of the more popular intervening variables is the reciprocal of x_i

$$y_i = \frac{1}{x_i} \; . \tag{6.1.6}$$

This popularity reflects the fact that many of the early structural optimization studies were performed on structures consisting of truss or plane-stress elements. The design variables in these studies were usually the cross-sectional areas of the truss elements and the thicknesses of the plane-stress elements. For statically determinate structures stress and displacements constraints are linear functions of the reciprocals of these design variables. For statically indeterminate structures, using the reciprocals of the design variables still proved to be a useful device in making the constraints more linear (see, for example, Storaasli and Sobieszczanski [3], and Noor and Lowder [4]). For the *reciprocal approximation* Eq. (6.1.5) becomes

$$g_R(\mathbf{x}) = g(\mathbf{x}_0) + \sum_{i=1}^{n} (x_i - x_{0i}) \frac{x_{0i}}{x_i} \left(\frac{\partial g}{\partial x_i}\right)_{\mathbf{x}_0} \; . \tag{6.1.7}$$

One of the attractive features of the reciprocal approximation, even for statically indeterminate structures, is that it preserves the property of scaling. That is, when the stiffness matrix is a homogeneous function of order h in the components of $\mathbf{x}$, the displacements are homogeneous functions of order $-h$ in the components of $\mathbf{x}$. For truss and membrane elements, $h = 1$ so that the displacements are homogeneous functions of the reciprocals of the design variables. If all the design variables are scaled by a factor, the displacement vector is scaled by the reciprocal of that factor. Therefore the reciprocal approximation is exact for scaling the design. Fuchs [5] has investigated the importance of the homogeneity property, and Fuchs and Haj Ali [6] have proposed a family of approximations that generalizes the reciprocal approximation to any order of homogeneity.

Another approximation, called the *conservative approximation* [7], is a hybrid form of the linear and reciprocal approximations which is more conservative than either. It is particularly suitable for interior and extended interior penalty function methods (see Section 5.7) which do not tolerate well constraint violations. To obtain the conservative approximation we start by subtracting the reciprocal approximation from the linear approximation

$$g_L(\mathbf{x}) - g_R(\mathbf{x}) = \sum_{i=1}^{n} \frac{(x_i - x_{0i})^2}{x_i} \left(\frac{\partial g}{\partial x_i}\right)_{\mathbf{x}_0} . \tag{6.1.8}$$

The sign of each term in the sum is determined by the sign of the ratio $(\partial g/\partial x_i)/x_i$ which is also the sign of the product $x_i(\partial g/\partial x_i)$. Contributions from design variables for which this product is negative make the reciprocal approximation larger (more positive) than the linear approximation, and vice versa. Since the constraint is expressed as $g(\mathbf{x}) \geq 0$, a more positive approximation is less conservative. The conservative approximation, g_C, is, therefore, created by selecting for each design variable the smaller (less positive) contribution

$$g_C(\mathbf{x}) = g(\mathbf{x}_0) + \sum_{i=1}^{n} G_i(x_i - x_{0i}) \left(\frac{\partial g}{\partial x_i}\right)_{\mathbf{x}_0} , \tag{6.1.9}$$

where

$$G_i = \begin{cases} 1 & \text{if } x_{0i}(\partial g/\partial x_i) \leq 0, \\ x_{0i}/x_i & \text{otherwise.} \end{cases} \tag{6.1.10}$$

Note that $G_i = 1$ corresponds to a linear approximation, and $G_i = x_{0i}/x_i$ corresponds to a reciprocal approximation in x_i.

The conservative approximation is not the only hybrid linear-reciprocal approximation possible. Sometimes physical considerations may dictate the use of linear approximation for some variables and the reciprocal for others, (see Haftka and Shore [8], and Prasad [9]). The conservative approximation, however, has the advantage of being concave (Exercise 1). If all the constraints are approximated by the conservative approximation, the feasible domain of the approximate optimization problem is convex (see Section 5.1.2). If we also approximate the objective function by a convex function, the approximate optimization problem is convex. Convex problems are guaranteed to have only a single optimum, and they are amenable to treatment by dual methods (see Section 9.2.2). In fact, a convex approximation $f_C(\mathbf{x})$ to the objective function, $f(\mathbf{x})$, is obtained by reversing the process for obtaining the conservative concave approximation. That is (Exercise 1),

$$f_C(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} F_i(x_i - x_{0i}) \left(\frac{\partial f}{\partial x_i}\right)_{\mathbf{x}_0} , \tag{6.1.11}$$

where

$$F_i = \begin{cases} x_{0i}/x_i & \text{if } x_{0i}(\partial f/\partial x_i) \leq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{6.1.12}$$

This process of using the conservative approximation for the constraints and the convex approximation for the objective function has been introduced by Braibant and

Fleury [10], and is known as convex linearization. In many papers and textbooks, the constraints are posed as $g(\mathbf{x}) \leq 0$ rather than $g(\mathbf{x}) \geq 0$. In this case, the conservative approximation is convex rather than concave (that is we use the form of Eqs. (6.1.11) and (6.1.12) also for the constraints). There are other conservative approximations (for example, see Prasad [11] or Woo [12]), but it is important to note that the one presented here, as well as the others, are not guaranteed to be conservative in an absolute sense (that is, we do not know that the approximation is more conservative than the exact constraint, $g_C(\mathbf{x}) \leq g(\mathbf{x})$). The approximation presented here is only more conservative than either the linear and reciprocal approximations.

Higher order approximations are also used occasionally. For example, the quadratic approximation, g_Q is obtained by including the quadratic terms in the Taylor series expansion

$$g_Q(\mathbf{x}) = g(\mathbf{x}_0) + \sum_{i=1}^{n}(x_i - x_{0i})\left(\frac{\partial g}{\partial x_i}\right)_{\mathbf{x}_0} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_{0i})(x_j - x_{0j})\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)_{\mathbf{x}_0} . \tag{6.1.13}$$

The reciprocal quadratic approximation g_{QR} is obtained by using the quadratic approximation in terms of the reciprocal design variables (Exercise 2),

$$\begin{aligned} g_{QR}(\mathbf{x}_0) = g(\mathbf{x}_0) &+ \sum_{i=1}^{n}\left(\frac{x_{0i}}{x_i}\right)\left(2 - \frac{x_{0i}}{x_i}\right)(x_i - x_{0i})\left(\frac{\partial g}{\partial x_i}\right)_{\mathbf{x}_0} \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{x_{0i}}{x_i}\right)\left(\frac{x_{0j}}{x_j}\right)(x_i - x_{0i})(x_j - x_{0j})\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)_{\mathbf{x}_0} . \end{aligned} \tag{6.1.14}$$

Example 6.1.2

Comparison of various approximations is demonstrated through the use of a simple

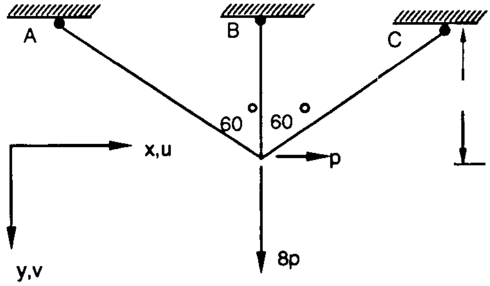

Figure 6.1.2 Three bar truss.

three bar truss shown in Figure 6.1.2. The horizontal force p can act either to the right (as shown) or to the left. The truss is designed subject to stress and displacement constraints with the design variables being the cross-sectional areas A_A, A_B, and A_C. Because of the symmetry of the truss and the arbitrary direction of the horizontal load we must have $A_A = A_C$. We examine the approximations to the constraint on the stress in member C, which requires that stress to be less than σ_0 both in tension and compression.

The stresses in the three members can be expressed in terms of the displacement components at the tip of the truss as

$$\sigma_A = E(v + \sqrt{3}u)/4l, \qquad \sigma_B = Ev/l, \qquad \text{and} \qquad \sigma_C = E(v - \sqrt{3}u)/4l \ .$$

From the horizontal equation of equilibrium

$$\frac{\sqrt{3}}{2} A_A(\sigma_A - \sigma_C) = p, \qquad \text{or} \qquad \frac{3EA_A}{4l} u = p \ .$$

Similarly, from the vertical equation of equilibrium

$$\frac{1}{2} A_A(\sigma_A + \sigma_C) + A_B\sigma_B = 8p, \qquad \text{or} \qquad \frac{Ev}{l}\left(A_B + \frac{A_A}{4}\right) = 8p \ ,$$

so that

$$u = 4pl/3EA_A, \qquad v = 8pl/E(A_B + 0.25A_A) \ ,$$

and

$$\sigma_C = p\left(-\frac{\sqrt{3}}{3A_A} + \frac{2}{A_B + 0.25A_A}\right) \ .$$

Assuming that member C is in tension, we may write the constraint function as

$$g = 1 - \frac{\sigma_C}{\sigma_0} = 1 - \frac{p}{\sigma_0}\left(-\frac{\sqrt{3}}{3A_A} + \frac{2}{A_B + 0.25A_A}\right) \ .$$

We now define normalized design variables

$$x_1 = A_A\sigma_0/p, \qquad x_2 = A_B\sigma_0/p \ ,$$

so that

$$g = 1 + \frac{\sqrt{3}}{3x_1} - \frac{2}{x_2 + 0.25x_1} \ .$$

We approximate g about the point $\mathbf{x}_0^T = (1, 1)$. The first derivatives are

$$\frac{\partial g}{\partial x_1} = \left(-\frac{\sqrt{3}}{3x_1^2} + \frac{0.5}{(x_2 + 0.25x_1)^2}\right)_{\mathbf{x}_0} = -0.2574 \ ,$$

$$\frac{\partial g}{\partial x_2} = \frac{2}{(x_2 + 0.25x_1)^2}\bigg|_{\mathbf{x}_0} = 1.28 \ .$$

and the second derivatives are

$$\frac{\partial^2 g}{\partial x_1^2} = \left(\frac{2\sqrt{3}}{3x_1^3} - \frac{0.25}{(x_2+0.25x_1)^3} \right)_{\mathbf{x}_0} = 1.0267 \ ,$$

$$\frac{\partial^2 g}{\partial x_1 x_2} = -\frac{1}{(x_2+0.25x_1)^3}\bigg|_{\mathbf{x}_0} = -0.512 \ ,$$

$$\frac{\partial^2 g}{\partial x_2^2} = -\frac{4}{(x_2+0.25x_1)^3}\bigg|_{\mathbf{x}_0} = -2.048 \ .$$

Using these derivatives and $g(\mathbf{x}_0) = -0.0227$ we can construct the following approximations

$$g_L = -0.0227 - 0.2574(x_1 - 1) + 1.28(x_2 - 1) \ ,$$

$$g_R = -0.0227 - 0.2574\left(1 - \frac{1}{x_1}\right) + 1.28\left(1 - \frac{1}{x_2}\right) = 1 + .2574/x_1 - 1.28/x_2 \ ,$$

$$g_C = -0.0227 - 0.2574(x_1 - 1) + 1.28\left(1 - \frac{1}{x_2}\right) \ ,$$

$$g_Q = g_L + 0.5134(x_1 - 1)^2 - 0.512(x_1 - 1)(x_2 - 1) - 1.024(x_2 - 1)^2 \ ,$$

$$g_{QR} = -0.0227 - 0.2574\left(2 - \frac{1}{x_1}\right)\left(1 - \frac{1}{x_1}\right) + 1.28\left(2 - \frac{1}{x_2}\right)\left(1 - \frac{1}{x_2}\right)$$
$$+ 0.5134\left(1 - \frac{1}{x_1}\right)^2 - 0.512\left(1 - \frac{1}{x_1}\right)\left(1 - \frac{1}{x_2}\right) - 1.024\left(1 - \frac{1}{x_2}\right)^2 \ .$$

All of these approximations have the correct value and correct derivatives at $\mathbf{x}_0 = (1,1)^T$. The two quadratic approximations also have the correct second derivatives at that point. The reciprocal approximations tend to one as the design variables tend to infinity. This corresponds to the stress in member C tending to zero as the cross-sectional areas tend to infinity. This correct physical behavior is not shared by the other approximations. Table 6.1.1 compares the predictions of the five approximations to the exact values when x_1 and x_2 vary between 0.75 and 1.25.

Table 6.1.1

x_1	x_2	g	g_L	g_R	g_C	g_Q	g_{QR}
0.75	0.75	-0.3635	-0.2783	-0.3635	-0.3850	-0.3422	-0.3635
1.00	0.75	-0.4227	-0.3426	-0.4493	-0.4493	-0.4066	-0.4209
1.25	0.75	-0.4205	-0.4070	-0.5008	-0.5137	-0.4070	-0.4280
0.75	1.00	0.0856	0.0417	0.0631	0.0417	0.0738	0.0915
1.25	1.00	-0.0619	-0.0870	-0.0741	-0.0871	-0.0549	-0.0639
0.75	1.25	0.3786	0.3617	0.3191	0.2977	0.3617	0.3919
1.00	1.25	0.2440	0.2974	0.2334	0.2334	0.2334	0.2435
1.25	1.25	0.1819	0.2330	0.1819	0.1690	0.1691	0.1819

The Table shows that the approximations based on reciprocal variables are more accurate than the approximations based on the actual variables, and in particular, they are exact when the two variables are scaled by the same factor (that is $\mathbf{x}$ is replaced by $\alpha\mathbf{x}$ where α is a scalar). The quadratic approximations are substantially more accurate than the three first-order approximations. The conservative approximation is not guaranteed to be more conservative than the second-order approximations, but usually, as in this example, it is. We see, however, that the price of this extra conservativeness is that it is the least accurate approximation.

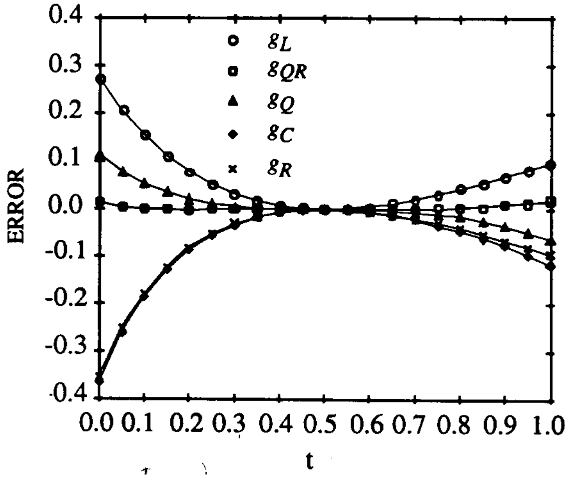

Figure 6.1.3 Comparison of constraint approximation errors.

The constraint approximations can also be used to check for errors in the derivatives used to construct them. This is done by calculating the exact constraint along a line in design space and plotting the error in the approximation along that line. A first order approximation must have a zero slope for the error curve at the nominal design, while a second-order approximation must also have zero curvature there. For example, let us compare the various approximations along the line

$$x_1 = 1.25 - 0.5t, \quad x_2 = 0.5 + 1.0t, \qquad 0 \le t \le 1 \ ,$$

where $t = 0.5$ represents the nominal design. Figure 6.1.3 shows the error as a function of t. It is seen that the first-order approximations indeed have zero slope at $t = 0.5$, while the second-order approximations also have zero curvature there. For this example, the reciprocal approximation is quite conservative, so that the conservative approximation is almost identical to it.• • •

The approximations covered so far are obtained by algebraically manipulating the constraint functions. In an effort to improve the quality of the approximations recent research efforts have concentrated on the extension of the concept of intermediate design variables to the concept of intermediate response quantities. The concept was introduced by Schmit and Miura [13] in 1976, but it was not applied until about ten years later (e.g., [14]). The approach seeks intermediate response quantities that are well approximated linearly. If the response quantities appearing in the constraint can be calculated inexpensively from the intermediate response, than we can have a nonlinear inexpensive and accurate approximation.

One of the most successful intermediate response approximation was proposed for stress constraints in structural design by Vanderplaats and coworkers (e.g., [15–17]). Vanderplaats argued that an approximation for member forces will be more accurate than the corresponding approximation for member stresses. This is expected because member forces change more slowly than member stresses when cross-sectional areas are changed. In particular, for a statically determinate truss, force in each of the members is constant, while member stresses are inversely proportional to member areas. This motivates the use of the member forces as intermediate response quantities.

Consider, for example, a typical stress constraint for a truss member of the form

$$g_i = 1 - \frac{\sigma_i}{\sigma_{\text{all}}} \geq 0 \; . \tag{6.1.15}$$

A common approximation for member stresses uses the reciprocal design variables, $x_i = 1/A_i$, where A_i is the cross-sectional area of the ith member. Using a linear approximation for the member forces, and then dividing by the cross-sectional area to obtain an approximation to the stress, as suggested by Vanderplaats, we obtain a constraint of the form

$$g_{LF\,i} = A_i - \frac{[F_i(\mathbf{A}_0) + \nabla^T F_i(\mathbf{A}_0)(\mathbf{A} - \mathbf{A}_0)]}{\sigma_{\text{all}}} \geq 0 \; . \tag{6.1.16}$$

This is linear in the cross-sectional area design variables. Note that for a statically determinate truss, where the gradient of the member forces with respect to the cross-sectional areas is zero, the approximation of Eq. (6.1.16) is a constant. Equation (6.1.16) has the dimension of area, and it should be nondimensionalized by dividing it by a reference area. A comparison of the performance of this linear force approximation with other approximations is given in Section 6.4.